Laboratory experiment

Study of free fall and measurement of the free-fall acceleration

1.1 Tasks

1. Measure the dependence of free-fall time on height for two steel balls of different diameters.
2. For both the balls, plot a graph of the height versus the time of free-fall, calculate the free-fall acceleration and its uncertainty and compare the results with the local value for Prague.

1.2 Theory

1.2.1 Newton's law of gravitation

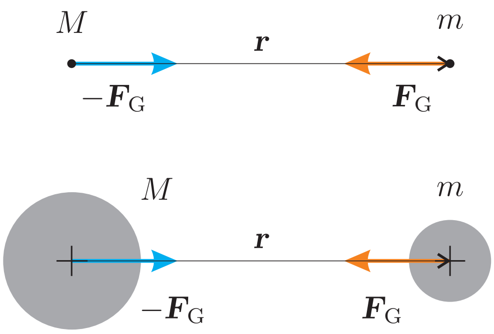


Figure 1.1: Regarding Newton's law of gravitation.

According to Newton's law of gravitation, the magnitude of the gravitational force exerted on two mass particles is directly proportional to the product of their masses and inversely proportional to the square of their distance. Using the formula, we can write

$$F_{\mathrm{G}} = G\frac{mM}{r^2}, \tag{1.1}$$

where m, M are masses of the individual particles, r is their mutual distance, and G is the so-called gravitational constant, whose value

$$G = (6.674\,08 \pm 0.000\,31) \cdot 10^{-11}\,\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2},$$

is determined experimentally.

The gravitational force is attractive one, if we introduce the gravitational force vector $\boldsymbol{F}_{\mathrm{G}}$, describing the force exerted by the particle M on a particle m, Eq. (1.1) can be rewritten as

$$\boldsymbol{F}_{\mathrm{G}} = -G\frac{mM}{r^2}\boldsymbol{r}_0 = -G\frac{mM}{r^3}\boldsymbol{r}, \tag{1.2}$$

where $\boldsymbol{r}$ is the position vector from the mass particle M towards the mass particle m, and $\boldsymbol{r}_0 = \boldsymbol{r}/r$ is corresponding unit vector. According to Newton's third law it applies for the force $\boldsymbol{F}'_{\mathrm{G}}$, exerted by the mass particle m on the mass particle M that $\boldsymbol{F}'_{\mathrm{G}} = -\boldsymbol{F}_{\mathrm{G}}$.

Equation (1.2) holds true exactly for mass particles. For extended bodies, the appropriate relation must be found by decomposing the volumes of the bodies into infinitesimal elements and "summing" the elementary contributions using the integral. It can be shown (laboriously but straightforwardly) that the relation (1.2) holds exactly only for spheres of masses M and m with spherically-symmetrically distributed mass, the position vector $\boldsymbol{r}$ being then defined by the geometric centers of the spheres. The relation (1.2) also holds for the force acting between a sphere and a mass particle.

1.2.2 Earth's gravitational field

In this paragraph we examine the motion of a body in the Earth's gravitational field, and for the sake of illustration we shall do so in several approximations.

Earth as a non-rotating sphere

Assume first that the Earth is a sphere of radius R_E, mass $M = M_\mathrm{E}$, with spherically symmetrically distributed mass, and assume first that it does not rotate along its axis. We will further assume that a body of mass m, whose motion we will investigate, is small enough relative to the Earth to be considered a mass particle, and that $m \ll M_\mathrm{E}$ holds. Neglecting the effect of the atmosphere[1], the only one force acting on the freely moving body will be the gravitational force $\boldsymbol{F}_\mathrm{G}$ described by Eq. (1.2).

The motion of the body can be calculated by substituting the gravitational force into Newton's second law, thus obtaining the equation of motion

$$m\boldsymbol{a} = \boldsymbol{F}_\mathrm{G} \quad \Rightarrow \quad m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -G\frac{mM_\mathrm{E}}{r^3}\boldsymbol{r} \quad \Rightarrow \quad \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + G\frac{M_\mathrm{E}}{r^3}\boldsymbol{r} = \boldsymbol{0}, \tag{1.3}$$

for the solution of which we need to know the initial conditions (for the position vector and its derivative – the velocity vector) at some time t_0. From the equation (1.3) it is immediately seen that the motion of a body moving freely in a gravitational field does not depend on its mass m.

The exact analytical solution of Eq. (1.3) is known[2]; however, it is too much complicated for the solution of many practical problems.

The complicatedness of solving Eq. (1.3) is caused by the nonlinear form of the functional dependence of the gravitational force Eq. (1.2) – its magnitude is not a constant (it decreases with the square of the distance r) and also its orientation changes from point to point (it is oriented towards the center of the sphere).

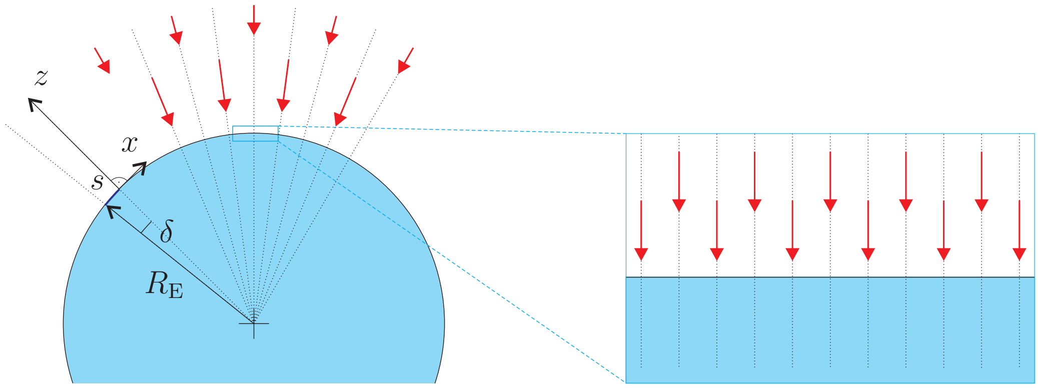
The situation is greatly simplified if we restrict ourselves to describing motion in a limited space near the Earth's surface. If we introduce local Cartesian coordinates at a given point on the Earth such that the positive direction of the z-axis points in the direction of the unit vector $\boldsymbol{r}_0$ and the z-axis is zero at the surface of the Earth, we can write for the z-component of the gravitational force vector that

$$F_{\mathrm{G}z} = -G\frac{mM_\mathrm{E}}{(R_\mathrm{E} + z)^2}. \tag{1.4}$$

This formula describes the decrease of the gravitational force with the distance z from the Earth's surface. For small distances z the formula (1.4) can be approximated employing the Taylor series

[1] The atmosphere manifests itself in two ways; on the one hand, it lifts the body by buoyancy and on the other hand, it hinders its motion by the action of the drag force.

[2] This is the so-called Kepler problem, the solution of which implies that bodies in the central force-field move along conic trajectories.



Central force-field

Uniform force-field

Figure 1.2: Approximation of the central field by a uniform field.

at point $z = 0$ as

$$F_{Gz} = -\frac{GmM_E}{R_E^2}\left[1 - 2\frac{z}{R_E} + 3\left(\frac{z}{R_E}\right)^2 - \dots\right]. \tag{1.5}$$

If we restrict ourselves to small heights z (compared to the radius of the Earth), we can neglect the terms in square brackets compared to one and write

$$F_{Gz} \approx -ma_g, \qquad \text{where} \qquad a_g \equiv \frac{GM_E}{R_E^2}, \tag{1.6}$$

where a_g is the magnitude of the free-fall acceleration (acceleration due to gravity) at the Earth's surface. If we substitute $M_E = 5.972\,19 \times 10^{24}$ kg and the Earth's equatorial radius $R_E = 6\,378.14$ km, we get $a_g = 9.7980\,\mathrm{m{\cdot}s^{-2}}$.

The difference between Eqs. (1.5) and (1.6) increases with the increasing height z; retaining the second term in square brackets in Eq. (1.5) shows that the acceleration due to gravity near the Earth's surface decreases by ca. $0.003\,\mathrm{m{\cdot}s^{-2}}$ per every kilometer.

Since the gravitational force is directed to the centre of the Earth, the verticals (local axes z) at different points on the surface of the Earth have a non-zero angle δ, for which

$$\delta = \frac{s}{R_E},$$

where s is the distance (arc length) between the given points on the Earth's surface (approximated as a sphere), see Fig. 1.2. Direct substitution into this formula shows that the angle between two verticals is ca. $0.54'$ per every kilometer of the distance, in other words, the nearby verticals are more or less parallel, see Fig. 1.2.

Therefore, using local Cartesian coordinates on the Earth's surface, we can replace the central gravitational field by a uniform field and replace the gravitational force (1.2) by (an approximate) relation

$$\boldsymbol{F}_g = m\boldsymbol{a}_g \qquad \text{where} \qquad \boldsymbol{a}_g = [0, 0, -a_g], \tag{1.7}$$

where a_g is introduced by Eq. (1.6).

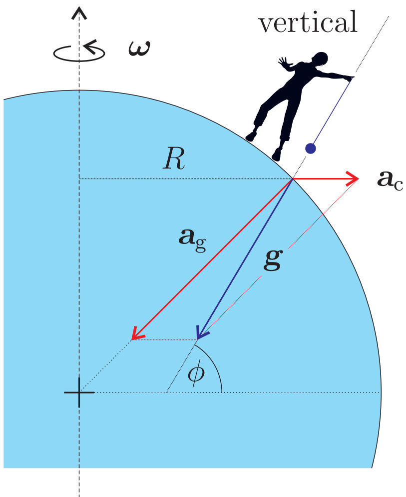


Figure 1.3: Free-fall acceleration.

Earth as a rotating sphere

Considering that the Earth rotates along its axis (with period $T = 1$ sidereal day[3]), then a reference frame fixed to the Earth is necessarily non-inertial, and to be able to to apply Newton's laws, we have to introduce inertial forces. In this case, we take into account the centrifugal force and the Coriolis force.

For the magnitude of the centrifugal force acting on a mass particle m in a rotating reference frame it applies

$$F_\mathrm{c} = ma_\mathrm{c} = m\omega^2 R, \tag{1.8}$$

where a_c is the magnitude of the centrifugal acceleration, ω is the magnitude of the rotation angular frequency, and R is the distance of the mass particle from the rotation axis. The centrifugal force is directed perpendicularly from the axis of rotation.

The Coriolis force acts only on bodies that are moving relative to the rotating reference frame and it applies

$$\boldsymbol{F}_\mathrm{C} = 2m\boldsymbol{v} \times \boldsymbol{\omega}, \tag{1.9}$$

where $\boldsymbol{v}$ is the velocity vector of the body relative to the rotating reference frame and $\boldsymbol{\omega}$ is the angular frequency vector[4] of the rotation of the reference frame.

The Coriolis force is a gyroscopic force (it does not change the magnitude of the velocity of bodies, only their direction), it is quite weak (in the reference frame of rotating Earth) and its effects will be neglected in this text[5].

The total force acting on a freely moving body described from the reference frame rigidly connected to the rotating Earth is therefore given by the vector sum of $\boldsymbol{F}_\mathrm{G} + \boldsymbol{F}_\mathrm{c}$.

The problem can be simplified, as in the previous case, if we restrict ourselves to describing motion in a limited space near the Earth's surface. The only difference is that the free-fall acceleration vector $\boldsymbol{g}$ is given by the sum of the gravitational and centrifugal acceleration vectors $\boldsymbol{g} = \boldsymbol{a}_\mathrm{g} + \boldsymbol{a}_\mathrm{c}$, it does not generally point to the centre of the Earth and its magnitude depends on the latitude ϕ (it increases with increasing latitude, with the maximum at the poles, and the minimum at the equator).

So if the Earth were a sphere of radius R_E, at the equator $R = R_\mathrm{E}$, and the centrifugal force would be directed radially from the centre of the Earth in the opposite direction to the force of gravity. So for the magnitude of the free-fall acceleration at the equator

$$g_\mathrm{equator} = a_\mathrm{g} - a_\mathrm{c} = \frac{GM_\mathrm{E}}{R_\mathrm{E}^2} - \frac{4\pi^2}{T^2} R_\mathrm{E} = 9.7641\,\mathrm{m \cdot s^{-2}},$$

which is by $0.0339\,\mathrm{m{\cdot}s^{-2}}$ less then the value at poles, where it holds $R = 0\,\mathrm{km}$ and the free-fall acceleration is equal to the acceleration due to gravity ($g_\mathrm{pole} = a_\mathrm{g}$).

In this case we can also introduce local Cartesian coordinates at different locations, but the local verticals do not have the radial direction (to the centre of the Earth), but a local direction of the free-fall acceleration (it can be easily determined with a plumb bob). Near the surface of the Earth in a limited space, we can then again consider the force field as uniform and write for the total force

$$\boldsymbol{F}_\mathrm{g} = m\boldsymbol{g} \qquad \text{where} \qquad \boldsymbol{g} = [0, 0, -g], \tag{1.10}$$

[3]One sidereal day is approximately equal to 23h 56m.

[4]This vector has the direction of the axis of rotation, its orientation is determined by the right hand rule.

[5]For example, it can be shown that the Coriolis force deflects a ball released in free fall from the Petřín tower (the Prague Eiffel tower), from a height of 60 m, by 6.5 mm to the east.

Earth as a rotating spheroid

The situation is actually a bit more complicated, as the Earth is slightly flattened due to its rotation at the poles (the Earth's polar radius is $R_\mathrm{p} = 6\,356.75\,\mathrm{km}$) and its shape resembles a spheroid. Therefore, the magnitude of the acceleration due to gravity at the Earth's surface is also a function of latitude. To calculate the magnitude of the free-fall acceleration at the sea level, an approximate empirical formula can be used[6]

$$g = g_\mathrm{e}(1 + \gamma_2 \sin^2\phi + \gamma_4 \sin^4\phi), \tag{1.11}$$

where $g_\mathrm{e} = 9.780\,327\,\mathrm{m{\cdot}s^{-2}}$ is the free-fall acceleration at the equator, ϕ is the latitude, and $\gamma_2 = 0.005\,279\,2$, and $\gamma_4 = 0.000\,023\,2$ are correction coefficients. From Eq. (1.11) it follows, that the free-fall acceleration at the equator $g_\mathrm{equator} \approx 9.78\,\mathrm{m{\cdot}s^{-2}}$, and at the poles $g_\mathrm{poles} \approx 9{,}83\,\mathrm{m{\cdot}s^{-2}}$. In Prague ($\phi = 50°\,06'$) then $g_\mathrm{Prague} \approx 9.81\,\mathrm{m{\cdot}s^{-2}}$. With altitude, the free-fall acceleration decreases, as shown above, at a rate of about $0.003\,\mathrm{m{\cdot}s^{-2}}$ per every kilometer.

The field in the limited space near the Earth's surface can be considered uniform, we can introduce the force vector in local Cartesian coordinates using the relation (1.10) and use Eq. (1.11) for the magnitude of the free-fall acceleration.

1.2.3 Motion in the Earth's gravitational field

It is easy to investigate the motion of a body (mass particle) in the Earth's gravitational field, adopting the previous assumptions and simplifications. Neglecting again the air drag, the equation of motion together with the relation for the gravitational force (1.10) can be written as

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{g} \qquad \Rightarrow \qquad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{g}. \tag{1.12}$$

As the vector $\boldsymbol{g}$ now (in a limited space) can be considered uniform, Eq. (1.12) can be easily integrated

$$\mathrm{d}\boldsymbol{v} = \boldsymbol{g}\,\mathrm{d}t \qquad \Rightarrow \qquad \int_{\boldsymbol{v}_0}^{\boldsymbol{v}} \mathrm{d}\boldsymbol{v}' = \int_0^t \boldsymbol{g}\,\mathrm{d}t' \qquad \Rightarrow \qquad \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{g}t + \boldsymbol{v}_0, \tag{1.13}$$

where $\boldsymbol{v}_0$ is the mass particle velocity at the time $t = 0$ (the initial condition). Integrating Eq. (1.13) the time-dependence of the position vector can be calculated as

$$\mathrm{d}\boldsymbol{r} = (\boldsymbol{g}t + \boldsymbol{v}_0)\,\mathrm{d}t \quad \Rightarrow \quad \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \mathrm{d}\boldsymbol{r}' = \int_0^t (\boldsymbol{g}t' + \boldsymbol{v}_0)\,\mathrm{d}t' \quad \Rightarrow \quad \boldsymbol{r} = \frac{1}{2}\boldsymbol{g}t^2 + \boldsymbol{v}_0 t + \boldsymbol{r}_0, \tag{1.14}$$

where $\boldsymbol{r}_0$ is the position vector of the mass particle at time $t = 0$ (the initial condition).

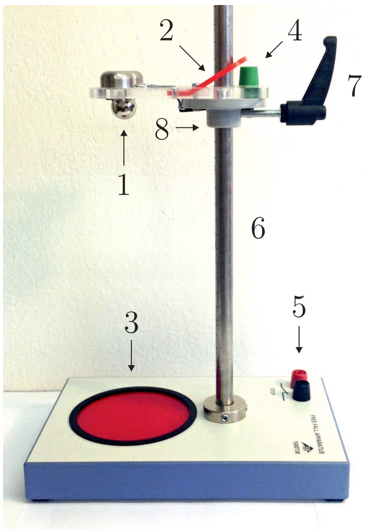


Figure 1.4: Free fall apparatus.

[6] The actual value of the free-fall acceleration differs slightly from this formula at different places on Earth, because the Earth is not a spheroid and the crust has different densities in different places.

An example: Vertical projectile motion

A projectile (mass particle) is launched from the position $\boldsymbol{r}_0 = [0, 0, h]$ (from the height of h) vertically with an initial velocity $\mathbf{v}_0 = [0, 0, v_0]$. As it holds $\boldsymbol{g} = [0, 0, -g]$, substitution into individual components of Eq. (1.14) results in

$$x(t) = 0, \qquad y(t) = 0, \qquad z(t) = -\frac{1}{2}gt^2 + v_0 t + h.$$

The time of free fall τ can be calculated by substituting $z = 0$ and solving the quadratic equation

$$h = \frac{1}{2}g\tau^2 - v_0\tau. \tag{1.15}$$

1.3 Experiment

Figure 1.5: Timer / counter.

The experiment works as follows. A steel ball is dropped from different heights h_i and the corresponding fall times τ_i are measured. A simple device called the free fall apparatus is used for the measurement, see Fig. 1.4. A steel ball $\boxed{1}$ is held in the top of the apparatus by a small neodymium magnet. When the trigger $\boxed{2}$ is pressed, the ball is released and falls onto the impact detector $\boxed{3}$. Information about the ball's release and impact is transmitted to the timer via the connecting cords $\boxed{4}$ and $\boxed{5}$. The ball holder is moved up and down on a centimeter-divided rod $\boxed{6}$, and a screw $\boxed{7}$ is used to release and fix the holder.

The timer/counter for measuring the ball drop time is shown in Fig. 1.5. The rotary switch $\boxed{1}$ selects the instrument mode, the ball drop time is measured in the position Δt_{AB}, the instrument allows to measure with a resolution of 0.1 s, 1 ms and 0.1 ms. Pressing the **RESET** $\boxed{2}$ button resets the display, the ball release contact is connected to the **IN START/COUNT** $\boxed{3}$ terminal, the ball impact sensor to the **IN STOP** $\boxed{4}$ terminal (the terminal colours must be respected). The device is switched on and off by a switch on the power cord.

1.4 Processing the measured data

For a sphere of a given diameter, the respective fall times τ_i are measured for different heights h_i. The measured pairs of values (τ_i, h_i) are approximated with a second degree polynomial employing the least squares method[7]

$$h = a_2\tau^2 + a_1\tau + a_0. \tag{1.16}$$

[7] For this purpose you can use the *Universal tool for plotting graphs* available at website *http://planck.fel.cvut.cz/praktikum/*.

Comparing Eqs. (1.15) and (1.16) shows that the free-fall acceleration corresponds to the coefficient multiplying the quadratic term and it holds

$$g = 2a_2, \qquad \sigma_g = 2\sigma_{a_2}.$$

The coefficient a_1 corresponds to the initial speed of the ball, and according to the free-fall apparatus' release mechanism design, its value should be close to zero.

The coefficient a_0 corresponds to the difference between the actual drop height of the ball and the value set on the bar with centimeter division. Since the drop height of the ball for a given line on the rod depends on the diameter of the ball, it is difficult to set the actual drop height on the rod. However, thanks to least squares processing, it is possible to proceed by setting the bottom edge of the ball holder ($\boxed{8}$, see Fig. 1.4) to the individual division lines of the rod. This does introduce a systematic error in the setting of the drop height, but this is not reflected in the result (the magnitude of the measured free-fall acceleration), but only in the value of the coefficient a_0.

1.5 Procedure

1. Choose one steel ball, measure its diameter and weight.
2. Using the free fall apparatus described in Sec. 1.3 measure the free-fall times τ_i for several (at least 10) heights h_i.
3. Using the procedure described in Sec. 1.4, calculate the magnitude of the free-fall acceleration and its uncertainty; compare the measured value with the local value for Prague.
4. Plot a graph of the dependence of the height of the fall on the time of the fall (approximate the measured values with the second-degree polynomial from which you have determined the free-fall acceleration).
5. Repeat the points 1–4 with a steel ball of different radius.

1.6 References

1. Jiří Bajer: Mechanika 1, *Univerzita Palackého v Olomouci*, Olomouc, 2004.
2. Jiří Bajer: Mechanika 2, *Univerzita Palackého v Olomouci*, Olomouc, 2004.
3. Kolektiv autorů: Výkladový slovník fyziky pro základní vysokoškolský kurz, *Prometheus*, 1999.

1.7 Appendix – Influence of the atmosphere

1.7.1 Buoyancy

In fluids, including the atmosphere – air – there is a buoyant force acting on bodies. It is equal to the weight of the fluid displaced by the volume of the body; according to Archimedes' law, it acts against the gravitational force, and for the sum of the forces acting on a body that is at rest relative to the fluid it applies

$$\boldsymbol{F}'_{\mathrm{g}} = \boldsymbol{F}_{\mathrm{g}} + \boldsymbol{F}_{\mathrm{b}} = m\boldsymbol{g} - \rho_0 V\boldsymbol{g} = (\rho - \rho_0)V\boldsymbol{g}, \tag{1.17}$$

where ρ_0 is the mass density of the fluid (atmosphere), ρ is the average density of the body, and V is its volume.

1.7.2 Drag

When a body moves in a fluid, a resistive force – drag – acts on it, which is a manifestation of the fluid's viscosity. In general, this force is a complex function of the velocity, shape and size of the body and the viscosity of the fluid. To calculate the magnitude of the drag force, we can use the approximate empirical Newton's formula

$$F_{\mathrm{d}} \approx \frac{1}{2} C \rho_0 S v^2, \tag{1.18}$$

where C is a coefficient related to the shape of the body (for a sphere moving with a "higher speed", it holds[8] $C \approx 0,5$), ρ_0 is the fluid density, S is the section area of the body, and v is the speed of the body with respect to the fluid.

We now investigate, as in paragraph 1.2.3, the free fall from a height h considering the atmospheric drag described by Eq. (1.18). For simplicity, we will assume zero initial velocity. For the z-component of the velocity vector, the following holds

$$m \frac{\mathrm{d}v_z}{\mathrm{d}t} = -mg + \frac{1}{2} C \rho_0 S v_z^2. \tag{1.19}$$

The term describing the drag force has the positive sign because the body moving in free fall moves in the negative direction of the z axis and the drag force has the opposite direction to the velocity vector, i.e., positive. Subsequently we get

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = -g + \frac{C\rho_0 S}{2m} v_z^2 \quad \Rightarrow \quad \frac{\mathrm{d}v_z}{-g + \frac{C\rho_0 S}{2m} v_z^2} = \mathrm{d}t \quad \Rightarrow \quad \int_0^{v_z} \frac{\mathrm{d}v_z'}{-g + \frac{C\rho_0 S}{2m} v_z'^2} = \int_0^t \mathrm{d}t' \quad \Rightarrow$$

$$-\sqrt{\frac{2m}{C\rho_0 g S}} \operatorname{atanh}\left(\sqrt{\frac{C\rho_0 S}{2mg}} v_z\right) = t \quad \Rightarrow \quad v_z = -\sqrt{\frac{2mg}{C\rho_0 S}} \tanh\left(\sqrt{\frac{C\rho_0 g S}{2m}} t\right). \tag{1.20}$$

The result (1.20) with respect to the properties of the hyperbolic tangent function shows that the magnitude of the velocity v_z increases from zero to the limiting value

$$|v_z|_{\max} = \sqrt{\frac{2mg}{C\rho_0 S}}.$$

If we employ the Taylor series approximation of the hyperbolic tangent function

$$\tanh x = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \dots,$$

we get

$$v_z = -gt + \frac{C\rho_0 S g^2}{6m} t^3 - \dots. \tag{1.21}$$

from which it can be seen that the body initially, as long as the magnitude of the drag force is small, moves with the constant acceleration g.

[8] The value of C for a given body shapes depends on the Reynolds number.

Integration of Eqs. (1.21) and (1.20) results in

$$z(t) = h - \frac{1}{2}gt^2 + \frac{C\rho_0 S g^2}{24m}t^4 - \cdots = h - \frac{2m}{C\rho_0 S}\ln\left[\cosh\left(\sqrt{\frac{C\rho_0 g S}{2m}}t\right)\right]. \tag{1.22}$$

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