Laboratory experiment

# Newton's second law of motion

## 1.1  Task

The aim of this experiment is to "check the validity" of Newton's second law of motion the following way:

1. Verify that the magnitude of the acceleration of a body is inversely proportional to its mass.

2. Verify that the magnitude of the acceleration of a body is directly proportional to the magnitude of the force acting on it.

## 1.2  Theory

### 1.2.1  Newton's laws of motion

The basic laws of dynamics, i.e., the laws describing the relationships between the forces acting on bodies and their motion, are the so-called Newton's laws of motion.

**Newton's first law – the law of inertia**

*An object at rest remains at rest, and an object that is moving will continue to move straight and with constant velocity, if and only if there is no net force acting on that object.*
In other words, if there are no forces $\boldsymbol{F}$ acting on the body (such a body is usually called free), the magnitude and direction of its velocity $\boldsymbol{v}$ are constant

$$\boldsymbol{F} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v} = \boldsymbol{const.}$$

Using this law we can define the inertial reference frame: *There is a reference frame in which a free body moves without changing its velocity. Such a reference frame is called inertial reference frame.*

**Newton's second law – the law of force**

*The acceleration of a body is directly proportional to the applied force, has the direction of the applied force and is inversely proportional to the mass of the body.* In other words,

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m}.$$

The mass $m$ is therefore a measure of the inertial effects of the body. In most cases, there are multiple forces acting on the body. These forces can be superposed (vector summed) into a single resultant force $\boldsymbol{F} = \sum_k \boldsymbol{F}_k$.

If we know the resultant of the forces acting on a body, we can calculate its acceleration according to the law of force, and from the basic equations of kinematics also its velocity and position as a function of time. From the mathematical point of view, the force law is generally a system of ordinary differential equations of second order (equation of motion) of the form

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \boldsymbol{F}\left(t, \boldsymbol{r}, \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right), \tag{1.1}$$

where $\boldsymbol{r} = \boldsymbol{r}(t)$ is the position vector of the body. The equation of motion (1.1) has a unique solution if it is supplemented with initial conditions (e.g. at time $t = 0$): $\boldsymbol{r}(0) = \boldsymbol{r}_0$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Thus, if a constant force $\boldsymbol{F}$ is applied to the body, we get for its acceleration $\boldsymbol{a} = \boldsymbol{F}/m$, and for the velocity and position vector

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \quad \Rightarrow \quad \boldsymbol{v} = \int \boldsymbol{a}\,\mathrm{d}t = \frac{\boldsymbol{F}}{m}t + \boldsymbol{C}, \quad \boldsymbol{v}(0) = \boldsymbol{C} = \boldsymbol{v}_0 \quad \Rightarrow \quad \boldsymbol{v} = \frac{\boldsymbol{F}}{m}t + \boldsymbol{v}_0,$$

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \quad \Rightarrow \quad \boldsymbol{r} = \int \boldsymbol{v}\,\mathrm{d}t = \frac{1}{2}\frac{\boldsymbol{F}}{m}t^2 + \boldsymbol{v}_0 t + \boldsymbol{C}, \quad \boldsymbol{r}(0) = \boldsymbol{C} = \boldsymbol{r}_0 \quad \Rightarrow \quad \boldsymbol{r} = \frac{1}{2}\frac{\boldsymbol{F}}{m}t^2 + \boldsymbol{v}_0 t + \boldsymbol{r}_0.$$

**Newton's third law – the action-reaction law**

*Two bodies exert forces on each other of equal magnitudes but opposite orientations, lying on a common line of force*

$$\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}.$$

Mathematically, the sum of these forces is zero, but, this does not mean that they cancel each other. Each acts on a different body.

## 1.2.2 Experiment

We will investigate Newton's second law of motion by means of an experiment, which is schematically depicted in Fig. 1.1.



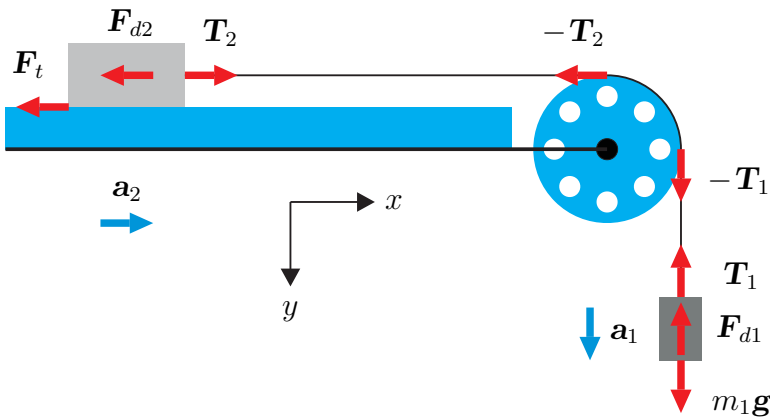Figure 1.1: Experimental set-up.

The gravitational force acting on a weight of mass $m_1$ through a pulley accelerates a body (glider) of mass $m_2$, which moves along a horizontal air-cushion track. The weight and the glider are connected by a thin silk thread, whose mass can be (by comparison with the masses of the other objects) completely neglected.

The equation of motion (for the component of the corresponding vectors in the direction of the $y$ axis) for a weight of mass $m_1$ has the form

$$m_1 a_1 = m_1 g - T_1 - F_{d1}, \tag{1.2}$$

where $a_1$ is the acceleration of the weight, $m_1 g$ is the gravitational force acting on the weight ($g$ is the acceleration due to gravity), $T_1$ is the force tensioning the vertical part of the thread, and $F_{d1}$ is the drag which air exerts on the moving weight.

The equation of motion (for the component of the corresponding vectors in the direction of the $x$ axis) for the horizontally moving glider of mass $m_2$ has the form

$$m_2 a_2 = T_2 - F_{d2} - F_t, \tag{1.3}$$

where $a_2$ is the acceleration of the glider, $T_2$ is the force tensioning the horizontal part of the thread, $F_{d2}$ is the drag which air exerts on the moving glider, and $F_t$ is the force of friction. As the glider moves on air-cushion, the force of friction is negligible and can be neglected ($F_t \approx 0$).

The equation of motion for the pulley has the form

$$J\varepsilon = rT_1 - rT_2 - M_b, \tag{1.4}$$

where $J$ is the pulley moment of inertia, $\varepsilon$ is its angular acceleration, $rT_1$ is the torque by which the vertical part of the thread rotates the pulley clock-wise, $rT_2$ is the torque by which the horizontal part of the thread brakes the pulley, and $M_b$ is the torque of the force of friction braking the rotation of the pulley. The design of the pulley used allows this torque to be neglected ($M_b \approx 0$).

As both the bodies are *rigidly* connected by the thread, $a_1 = a_2 = a$. The thread on the pulley does not slip, so $\varepsilon = a/r$. Adding Eq. (1.2) and (1.3) and substituting for $T_1 - T_2$ from the relation (1.4) gives the equation of motion

$$\left(m_1 + m_2 + \frac{J}{r^2}\right) a = m_1 g - F_{d1} - F_{d2}. \tag{1.5}$$

The term $J/r^2$ has the dimension of mass and it represents equivalent mass of the pulley $m_{\text{eq}}$; in this particular case, $m_{\text{eq}} = 2.2$ g. This mass can be included into the mass of the glider so that Eq. (1.5) becomes

$$(m_1 + m_2') a = m_1 g - F_{d1} - F_{d2}, \tag{1.6}$$

where $m_2' = m_2 + m_{\text{eq}} = m_2 + J/r^2$.

Next, we estimate the drag force of the environment acting on the moving bodies. We use Newton's formula

$$F_d \approx \frac{1}{2} C \rho S v^2, \tag{1.7}$$

where $C$ is the drag coefficient depending on the body's shape ($C \approx 0.5$ for a sphere, $C \approx 1$ for a flat plate, $C \approx 0.03$ for a drop-shaped body), $S$ is the cross-sectional area, and $\rho$ is the density of the fluid in which the object moves. The drag force always acts in the direction opposite to the object's velocity vector.

Assume for a moment that the drag forces $F_{d1}$ and $F_{d2}$ are so small that they can be neglected compared to the gravitational force $m_1 g$. From Eq. (1.6) we can calculate the acceleration

$$a = \frac{m_1 g}{m_1 + m_2'}.$$

The velocity of the motion with constant acceleration from rest is a function of the distance $s$ and it can be expressed by the well-known formula[1]

$$v = \sqrt{2as}.$$

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[1]It can be obtained by eliminating the time from relations $s = at^2/2$ and $v = at$.

Putting $m_1 = 0.02$ kg (the maximum weight by which the pulley can be loaded), $m_2 \approx 0.2$ kg (corresponding approximately to the mass of the glider), $s \approx 1$ m, $g = 9.81$ m/s$^2$, we get for the maximum velocity of the weight and the glider $v_{\max} \approx 1.3$ m/s. Cross-sectional area of the weight $S_1 \approx 1.6 \times 10^{-4}$ m$^2$, cross-sectional area of the glider $S_2 \approx 4.2 \times 10^{-4}$ m$^2$, the air density $\rho \approx 1.2$ kg/m$^3$. Putting all these values into Eq. (1.7), together with $C \approx 1$ results in the maximum drag force $F_{d1} + F_{d2} \approx 6.1 \times 10^{-4}$ N. This force decreases the action of the gravitational force $m_1 g$ of the accelerating weight by only ca. 0.3 %, so that it can be, within this experiment, neglected.

Thus, the equation of motion for the weight and the glider moving on the air-cushion track has the form

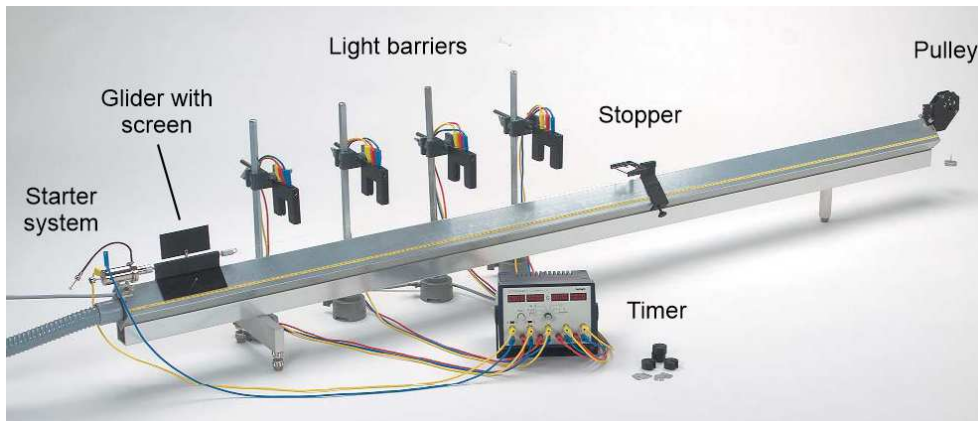$$(m_1 + m_2')a = m_1 g. \tag{1.8}$$

## 1.3   Instructions



Figure 1.2: Experimental set-up.

The aim of this measurement is to "verify the validity" of the force law, i.e., to show that the acceleration of a body is directly proportional to the applied force and inversely proportional to its mass.

In order to eliminate frictional forces, the accelerated body is represented by a glider moving on am air-cushion along a horizontal path.

The accelerating force is realized by the weight which is connected to the glider via a pulley by a thin thread. Since this force is constant, the acceleration of the glider is also constant and for its position on the track we can write

$$s = \frac{1}{2}at^2 + v_0 t + s_0.$$

The acceleration $a$ of the glider is measured as follows. At time $t = 0$ the starter system releases the glider, which from rest starts to move with constant acceleration along the track. At times $t_{1,2,3,4}$, a screen placed on the glider shades the light barriers at distances $s_{1,2,3,4}$ from the launcher. Using the least squares method (see paragraph 1.4), we approximate the measured values $t_i, s_i$ with the parabola $s = k_2 t^2 + k_1 t + k_0$, where the coefficient $k_2$ corresponds to half of the glider acceleration, $k_2 = a/2$, the coefficient $k_1$ corresponds to the initial velocity of the glider (it is not always possible to ensure exactly zero initial velocity) and $k_0$ corresponds to the launch position of the glider.

### 1.3.1   Experimental set-up

The experimental set-up for the "verification of the validity" of the force law is shown in Fig. 1.2.

Check that the starter system at the end of the air-cushion track is turned with the movable end out of the track. If not, loosen the clamping screws and rotate it. Connect the starter system

to the timer terminals, see Fig. 1.3, marked **START**, with the red terminal on the starter system to the yellow terminal on the timer.

Connect the light barriers to timer terminals **1-4** so that the corresponding colours match each other.

Switch the timer rotary mode switch to the position **1**. Switch both slide switches to the rightmost position. In this mode and wiring, the individual timer displays show the times from the glider launch to the respective light barriers shading by the screen attached to the glider. The measured times are reset by pressing the **RESET** button.
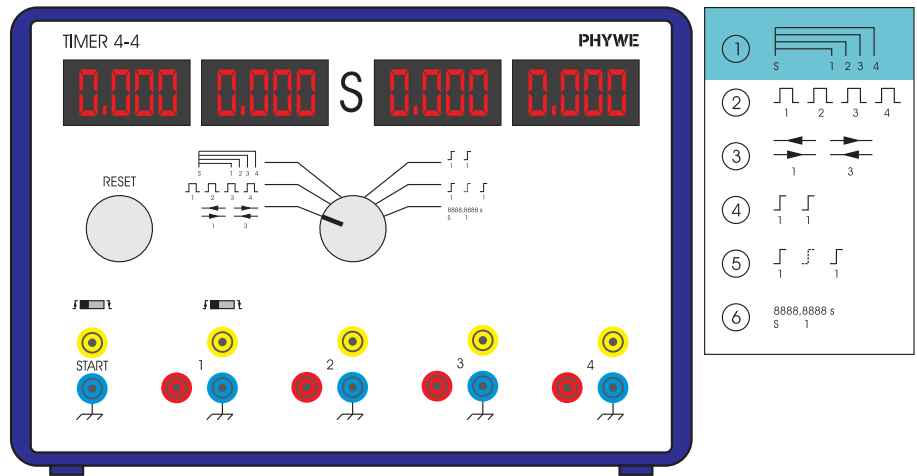


Figure 1.3: Timer.

Attach the plug with magnet (for the starter system) to the glider on one side and the plug with hook for attaching the thread on the other side. Place a stopper with a braking rubber band on the air track in such a distance to stop the glider before the weight hits the floor.

Place the light barriers evenly along the air track and measure their distances from the starter system. This is best done as follows. Attach the glider to the starter system and read its position on the scale along the air track. Switch the timer to the position **2** (in this mode, the display of each light barrier shows the time from cover to uncover), move the glider to the vicinity of the light barrier, and see at what position the timer starts (and stops). At this point, read the glider position. Remember to switch the timer back to the position **1** after reading the positions of all light barriers.

Turn on the blower, set the airflow to position **3-4**. Check that the glider can move freely along the air track and that it does not move spontaneously after being brought to rest. If it does, adjust the air track to the horizontal position as best you can. Check this setting from time to time during the measurements, especially when loading the glider.

## 1.3.2 Procedure

**Acceleration of a body for constant force and different masses**

1. Set up and adjust the experiment according to the instructions described in the previous paragraph.

2. Place one 10 g weight on the holder attached to the thread and measure the total mass $m_1$ (accelerating weight). Measure the mass of the glider $m_2$ (including the screen, the magnet for the starter system and the hook for attaching the thread).

3. Pull the starter and attach the glider to it, **check that the thread is stretched over the pulley** (sometimes it falls off).

4. Use the **RESET** button to reset the timer, release the glider with the wire trigger and read the glider's passage times through the light barriers on the timer. Make the measurement at least 5×.

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5. Place **symmetrically** two weights of 10 g each on the glider and continue with the measurement in step 3.

6. For each inertial mass $(m_1 + m_2')_i$, calculate the acceleration of the glider $a_i$ from the measured data, see the first paragraph of Instructions.

7. Use these values to calculate the force accelerating the glider (see paragraph Calculations – Method of reduction) and compare it with the force $F = m_1 g$.

8. Plot a graph comparing the theoretical acceleration versus the glider mass

$$a = \frac{m_1 g}{m_1 + m_2'}$$

together with the measured accelerations $a_i$ for the individual masses $(m_1 + m_2')_i$.

**Acceleration of a body for different forces and constant mass**

Contrary to the previous measurement, the difference is that now the total mass $m_1 + m_2'$ is constant.

1. Place one 10 g weight on the holder attached to the thread and measure the total mass $m_1$ (accelerating weight). Determine the mass of the glider $m_2$ (including the screen, the magnet for the starter system, and the hook for attaching the thread), on which you also place **symmetrically** 20 one-gram weights.

2. Pull the starter and attach the glider to it, **check that the thread is stretched over the pulley** (occasionally it will fall off).

3. Use the **RESET** button to reset the timer, release the glider with the wire trigger and read the glider's passage times through the light barriers on the timer. Make the measurement at least 5×.

4. Transfer two one-gram weights (one from each side) from the glider to the holder with the accelerating weight and continue with the measurement in step 2.

5. For each accelerating weight $(m_1 g)_i$, calculate the acceleration of the glider $a_i$ from the measured data, see the first paragraph of Instructions.

6. Use these values to calculate the inertial mass $m_1 + m_2'$ and compare it with the actual value.

7. Plot a graph comparing the theoretical acceleration versus the accelerating force $m_1 g$.

$$a = \frac{m_1 g}{m_1 + m_2'}$$

together with the measured accelerations $a_i$ for individual values $(m_1 g)_i$.

## 1.4 Calculations

### 1.4.1 Least squares method

You have several options for approximating the measured values with a second degree polynomial.

**Planck**

You can use the script *Universal tool for plotting graphs* available at server *http://planck.fel.cvut.cz/praktikum/*.

**Matlab**

In Matlab, you can use the command *polyfit*. Example:

```
% measured data
x = [0,  0.1, 0.2, 0.3, 0.4]; % independent variable
y = [0, 0.05, 0.2, 0.4, 0.8]; % dependent variable

n = 2; % degree of the polynomial

p = polyfit(x,y, n); % calculates the polynomial's coefficients

% acceleration: the first element represents the coefficient of
% the highest power and the acceleration is twice its value
a = 2*p(1)
```

**Maple**

You can use the command *fit* from the library *stats*. Example:

```
> with(stats): # initialization of the library

> dx := [0,  0.1, 0.2, 0.3, 0.4]; # independent variable
> dy := [0, 0.05, 0.2, 0.4, 0.8]; # dependent variable

> # returns the approximation function, here, y=a*x^2+b*x+c
> fit[leastsquare[[x,y], y=a*x^2+b*x+c]]([dx, dy]);
```

## 1.4.2   Method of reduction

For several total masses $m_i = (m_1 + m_2')_i$, $i = 1 \ldots N$ (accelerating weight + glider + correction) we measured and calculated the corresponding accelerations of the glider $a_i$. From the relation (1.8) we have to calculate the accelerating force $F = m_1 g$. The most straightforward method in this case is probably the *method of reduction*. The principle is very simple.

1. For the individual measured accelerations we calculate the corresponding accelerating forces $F_i = m_i a_i$.

2. The estimate of the accelerating force is calculated as the mean value for the individual measured and calculated values $F_i$, the uncertainty of this estimate is expressed by the empirical standard deviation

$$\overline{F} = \frac{1}{N} \sum_{i=1}^{N} F_i, \quad s_{\overline{F}} = \sqrt{\frac{\sum_{i=1}^{N}(F_i - \overline{F})^2}{N(N-1)}}.$$

We can proceed in a similar way in the second case, where for different accelerating forces $F_i = (m_1 g)_i$ we have measured and calculated the corresponding accelerations of the glider $a_i$. From the relation (1.8) we calculate the corresponding inertial masses $m_i = F_i/a_i$ and from these values we then calculate the mean value and estimate its standard deviation.

## 1.5    References

1. Jiří Bajer: Mechanika 1, *Univerzita Palackého v Olomouci*, Olomouc, 2004.

2. Jiří Bajer: Mechanika 2, *Univerzita Palackého v Olomouci*, Olomouc, 2004.

3. Semyon G. Rabinovich: Measurement Errors and Uncertainties: Theory and Practice, *Springer Science and Media, Inc.*, 2005.

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