Laboratory experiment

Determination of shear modulus and moment of inertia by dynamic method

1.1 Tasks

1. Measure the shear modulus of a steel string.
2. Determine the moment of inertia of a rotor of an electric motor by the method of torsional oscillations.

1.2 Introduction

1.2.1 Moment of inertia

Let's have a rigid body that can rotate around a fixed axis O, see Fig. 1.1. Such a body can only perform a rotational motion around the axis O and its position is thus completely determined by means of the angle of rotation. Imagine that the body is made up of a total of N point particles of mass m_i, where $i = 1, \ldots, N$. The total kinetic energy of this system can be thus calculated as

$$E_\mathrm{k} = \sum_{i=1}^{N} E_{\mathrm{k}i} = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2, \tag{1.1}$$

where v_i is the velocity magnitude of the i-th point particle. As we assume that the body is rigid, the individual material point particles that make it up do not change their position relative to each other. If the rigid body rotates with an angular velocity ω, its individual points move along circular trajectories and the following applies for their circumferential velocities

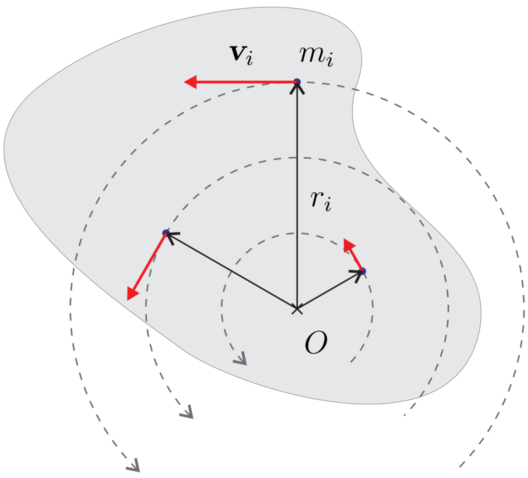

Figure 1.1: A rotating rigid body.

$$v_i = \omega r_i, \tag{1.2}$$

where r_i is the distance of a given point from the rotation axis O. If we substitute the relation (1.2) into (1.1) we obtain the relation for the total (rotational) kinetic energy of the rigid body as

$$E_\mathrm{k} = \frac{1}{2} \sum_{i=1}^{N} m_i \omega^2 r_i^2 = \frac{1}{2} \left(\sum_{i=1}^{N} m_i r_i^2 \right) \omega^2 = \frac{1}{2} J \omega^2, \tag{1.3}$$

where the quantity

$$J = \sum_{i=1}^{N} m_i r_i^2, \tag{1.4}$$

only depends on the mass distribution of the body with respect to the rotation axis and it does not depend on the angular velocity of the body. The quantity J is called *the moment of inertia* with respect to the given axis. For different axes of rotation, a given body generally has different moments of inertia.

The moment of inertia plays a key role in the dynamics of rotational motion and expresses the degree of inertia of the body in rotational motion, similarly as the (inertial) mass of the body in the translational motion. For practical calculations of the moment of inertia of a body with a continuously distributed mass, we can replace the individual material point particles in the relation (1.4) with the masses of elementary volumes $m_i \to \mathrm{d}m = \rho \mathrm{d}V$, where ρ is the density at a given point, replace the sum with a volume integral and write

$$J = \int_V \rho r^2 \mathrm{d}V, \tag{1.5}$$

where r represents the distance of element $\mathrm{d}V$ from the rotation axis.

Example: Moment of inertia of a cylinder

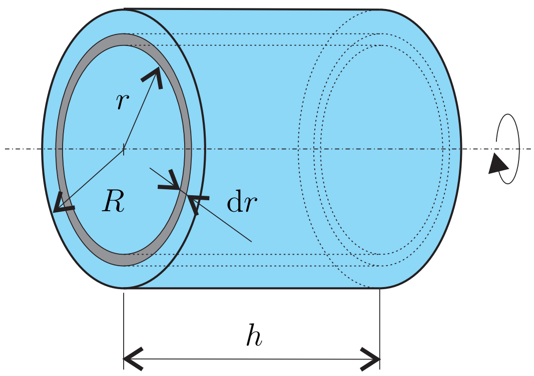

Figure 1.2: Calculation of moment of inertia of a cylinder.

The calculation of the moment of inertia is illustrated by the example of a homogeneous cylinder with radius R and height h for the axis of rotation identical to its geometric axis, see Fig. 1.2.

To avoid calculating the multiple (triple) integral, we express the volume element $\mathrm{d}V$ using a suitable single variable, here, the appropriate variable is r. Geometrically, the element $\mathrm{d}V$ is a cylindrical shell with the height h, the radius r and the wall thickness $\mathrm{d}r$, see Fig. 1.2. The volume of this element can be found either as a differential of the volume of the cylinder

$$V = \pi r^2 h \qquad \Rightarrow \qquad \mathrm{d}V = 2\pi h r \mathrm{d}r,$$

or by "straightening" the shell to a thin block with sides $h, 2\pi r, \mathrm{d}r$. The integration over the entire volume results in

$$J = 2\pi\rho h \int_0^R r^3 \,\mathrm{d}r = \frac{\pi \rho h R^4}{2}.$$

The cylinder density can be expressed as

$$\rho = \frac{m}{V} = \frac{m}{\pi r^2 h},$$

so the moment of inertia can be written as

$$J = \frac{1}{2} m R^2. \tag{1.6}$$

1.2.2 Shear elasticity

Let's have a prism with height h and bases with area S. If oppositely oriented tangential forces of the same magnitude F act on the upper and lower bases, shear stress and deformation of the prism occur. The side walls are skewed by the shear angle $\gamma \approx \tan\gamma = u/h$, see Fig. 1.3.

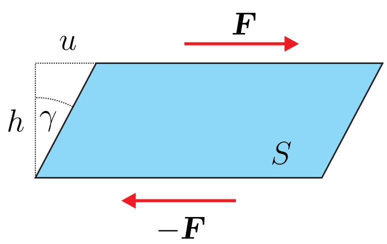

Figure 1.3: Shear stress.

The tangential stress τ has the magnitude

$$\tau = \frac{F}{S}.$$

If the height of the prism h is small enough, bending of the prism can be neglected and the Hooke's law[1] applies for the shearing angle γ as

$$\gamma = \frac{\tau}{G}, \tag{1.7}$$

where the proportionality constant G is called the shear modulus and it is a material constant, the value of which for a given material can be found in the tables, see, e.g., Table 1.1 on page 6.

1.2.3 Torsion elasticity

Shear elasticity is manifested, among other things, by torsion, or torsional loading of bodies. Consider a rod (string) of a circular cross-section, length l and radius a, which we subject to torsional stress with a torque of magnitude M, see Fig. 1.4. The volume of the rod can be divided into elementary cylindrical shells of height l, radius r and thickness dr. The individual shells are loaded with the tangential stress τ and they shear $\gamma = \tau/G$ according to Hooke's law. Rotation angle φ is the same throughout the cross section, so $u = r\varphi = l\gamma$. By substituting into Hooke's law, we get

$$\tau = G\frac{r\varphi}{l},$$

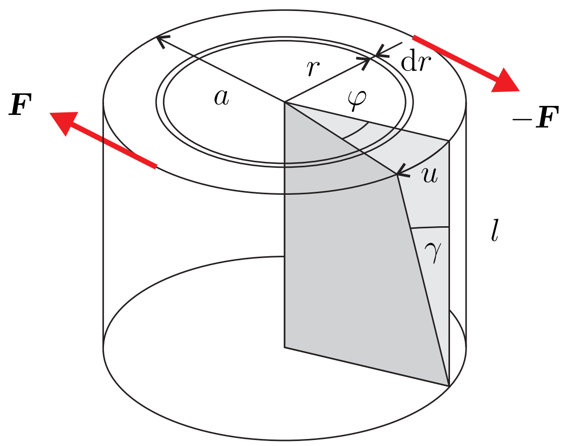

Figure 1.4: Torsion stress.

which means that the tangential stress increases linearly with the distance r from the center of the rod. An elementary torque acts on the elementary intermediate ring (upper base of the cylindrical shell) with area d$S = 2\pi r$dr which reads

$$\mathrm{d}M = r\mathrm{d}F = r\tau\mathrm{d}S = 2\pi G\frac{r^3\varphi}{l}\mathrm{d}r$$

and a total torque acts on the entire base

$$M = 2\pi G\frac{\varphi}{l}\int_0^a r^3\mathrm{d}r = \frac{\pi a^4 G}{2l}\varphi = k_\mathrm{T}\varphi, \tag{1.8}$$

where $k_\mathrm{T} = \pi a^4 G/2l$ is called the torsional stiffness[2] of the rod (string).

[1]This empirical rule also applies only if the tangential stress does not become too high.

[2]Note the strong dependence of the torsional stiffness on the radius. This property is employed in sensitive torsional scales with very thin fibres allowing to achieve measurable rotations even at very small torques. Torsional scales were used, for example, by Coulomb to determine the force acting between charges and by Cavendish to determine the gravitational constant.

1.2.4 Torsion pendulum

On a string of length l and diameter d we hang a body (flywheel) with a known moment of inertia J with respect to the axis of the suspension, twist the string by an angle φ_0 and release. The body on the string begins to perform *torsional oscillations* - we built the so-called *torsion pendulum.*

In order to twist the string by an angle φ, there must be a torque $M = k_{\mathrm{T}}\varphi$, acting on it, see Eq. (1.8). According to Newton's Third law, torque $M' = -M = -k_{\mathrm{T}}\varphi$ is the reaction to the torque M, which tries to return the string to the original position, so that we can write the following equation of motion for the body hung on the string

$$J\ddot{\varphi} = M' \qquad \Rightarrow \qquad \ddot{\varphi} + \omega_0^2\varphi = 0, \tag{1.9}$$

where

$$\omega_0 = \sqrt{\frac{k_{\mathrm{T}}}{J}} = \sqrt{\frac{\pi d^4 G}{32 l J}} \tag{1.10}$$

is the angular frequency of the torsional oscillations, as we can recognize the equation of linear harmonic oscillator in Eq. (1.9). For one swing period[3] T_{k} of the torsion pendulum we get

$$T_{\mathrm{k}} = \frac{\pi}{\omega_0} = \sqrt{\frac{32\pi l J}{d^4 G}}. \tag{1.11}$$

The relation (1.11) can be employed to determine the moment of inertia of the hung body (if we know the shear modulus of the string), or, if we know the moment of inertia of the hung body, we can rewrite Eq. (1.11) into the form

$$G = \frac{32\pi l J}{d^4 T_{\mathrm{k}}^2}, \tag{1.12}$$

which is suitable for the determination of the shear modulus of the string material.

1.3 Procedure

1.3.1 Determination of shear modulus

1. Measure the length of the string with a steel rule.
2. Measure the diameter of the string with a micrometer at least 10×.
3. Measure the diameter of cylindrical plate serving as a flywheel with a caliper. Its weight is indicated on it. Calculate its moment of inertia using the relation (1.6).
4. Hang the cylindrical plate on the string. Turn it by angle of approx. $60° - 90°$, release it and start to measure the swing period T_{k} of the torsional oscillations.
5. Use the method of limitations for the determination of the swing period (see Appendix 1.4).
6. Determine the standard uncertainty of the swing period $u(T_{\mathrm{k}})$. Substitute the swing period T_{k} into the formula (1.12) and calculate the shear modulus G of the string material.
7. Calculate the combined standard uncertainty of the shear modulus of the string material.

[3] One entire cycle duration T consists of two swings (back and forth), so that for one swing period T_{k} we get $T_{\mathrm{k}} = T/2$.

1.3.2 Measurement of the moment of inertia of a rotor of an electric motor

1. On a string whose parameters you know from the previous measurements, hang the body whose moment of inertia you want to measure.
2. Let the system perform torsional oscillations and use the method of limitations (see Appendix 1.4) to determine the swing period T_k of the system.
3. Use the swing period T_k in Eq. (1.12) to determine the moment of inertia J of the examined body.
4. Calculate the combined standard uncertainty of the moment of inertia of the examined body.

1.4 Appendix – Method of limitations

This method is suitable for measuring periodic events, in particular if the repetition period is large. Its main advantage is that it is possible to achieve theoretically arbitrary accuracy without laboriously counting the number of periods.

It is enough to know the estimate of the maximum error, that we can make when measuring one period of the studied process. Let's demonstrate the method on the example of measuring the swing period of the torsion pendulum.

The swing period of the used torsion pendulum is ca. 5 seconds.

We choose the duration of 10 swings as the elementary period and measure it by a stopwatch:

$$10T_\mathrm{k} = 52.8\,\mathrm{s}.$$

We estimate the value of the maximum measurement error we have made as 0.4 s (which depends on the used stopwatch and experimenter's reaction time) and thus we obtain the duration of 10 swings in interval

$$52.4\,\mathrm{s} < 10T_\mathrm{k} < 53.2\,\mathrm{s}.$$

For the duration of 20 swings we can expect the interval

$$104.8\,\mathrm{s} < 20T_\mathrm{k} < 106.4\,\mathrm{s}.$$

As this interval $(106.4 - 104.8 = 1.6\,\mathrm{s})$ is shorter than the duration of one swing $T_\mathrm{k} \approx 5.24\,\mathrm{s}$, we can start the stopwatch at the beginning of any swing and *without the swings counting* to stop it at the end of the swing which ends after the time of 104.8 s. The stopwatch may show the time, e.g., 105.4 s. Thus we find out that the uncertainty interval for the duration of 20 swings is

$$105\,\mathrm{s} < 20T_\mathrm{k} < 105.8\,\mathrm{s}.$$

From this, for the duration of 100 swings we get

$$525\,\mathrm{s} < 100T_\mathrm{k} < 529\,\mathrm{s}.$$

Also in this case the difference $(529 - 525 = 4\,\mathrm{s})$ is smaller than the duration of one swing and thus when reading the time of the end of the swing after 525 s we obtain the duration of $100T_\mathrm{k}$. If it is, e.g., 527.3 s, we know that the duration of 100 swings is in interval

$$526.9\,\mathrm{s} < 100T_\mathrm{k} < 527.7\,\mathrm{s},$$

and from here, we immediately get

$$5.269\,\mathrm{s} < T_\mathrm{k} < 5.277\,\mathrm{s}.$$

The expected value of the swing period is therefore with a high probability anywhere in the interval $\pm 0.004\,\mathrm{s}$ around the calculated value, so the standard uncertainty (determined by method of type B) can be estimated as $u(T_\mathrm{k}) \approx (0.004/\sqrt{3})\,\mathrm{s} = 0.0023\,\mathrm{s}$.

It is obvious that with this procedure we can achieve great accuracy for long enough periodic processes. A necessary condition is the choice of only such multiples of the elementary period that the uncertainty limits are smaller than the measured period.

1.5 Appendix – Selected properties of some materials

Material	E [10^{10} Pa]	G [10^{10} Pa]	k -
Aluminium	7.07	2.64	0.34
Copper	12.3	4.55	0.35
Lead	1.6	0.56	0.44
Diamond	112	52	0.1
Zinc	9.0	3.6	0.25
Iron α	21.2	8.2	0.29
Steel	20-21	7,9-8,9	0,25-0,33
Steel (1% C)	21.0	8.1	0.29
Welding steel	20.4	7.9	0.29
Bronze	9.7-10.2	3.3-3.7	0.34-0.40
Phosphor bronze	12.0	4.36	0.38
Brass	9.9	4.2	0.37
Duralumin	7.25	2.75	0.34
Plexiglass	0.33	0.12	0.35

Table 1.1: Young's modulus E, shear modulus G, and Poisson's constant k for selected materials at room temperature.

1.6 References

1. Michal Bednařík, Petr Koníček, Ondřej Jiříček: Fyzika I a II – Fyzikální praktikum, *[skriptum], Vydavatelství ČVUT*, Praha, 2003.
2. Jiří Bajer: Mechanika 2, *Univerzita Palackého v Olomouci*, Olomouc, 2008.
3. Jiří Bajer: Mechanika 3, *Univerzita Palackého v Olomouci*, Olomouc, 2012.

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(adapted from the original text [1])