Laboratory experiment

Determination of the specific charge of the electron

1.1 Task

From the curvature of the paths of electrons moving in a magnetic field, determine the specific charge of the electron.

1.2 Theory – Motion of a charged particle in an electric and magnetic field

1.2.1 The Lorentz force law

If a charged particle with charge q moves with velocity $\mathbf{v}$ in an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, there is so-called *Lorentz force* acting on it which can be mathematically described as

$$\boldsymbol{F} = q\left[\boldsymbol{E} + (\mathbf{v} \times \boldsymbol{B})\right]. \tag{1.1}$$

If we limit ourselves to non-relativistic speeds $v = |\mathbf{v}| \ll c$, where c is the vacuum speed of light, we can write the equation of motion for the particle as

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = q\left[\boldsymbol{E} + (\mathbf{v} \times \boldsymbol{B})\right], \tag{1.2}$$

where m is its mass, and $\boldsymbol{r}$ is the position vector. The solution of Eq. (1.2) can be found provided the initial conditions are known at a given (say, zero) time in form

$$\boldsymbol{r}(t=0) = \boldsymbol{r}_0, \qquad \mathbf{v}(t=0) = \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_{t=0} = \mathbf{v}_0. \tag{1.3}$$

1.2.2 Motion of a charged particle in a uniform electric field

If it holds $\boldsymbol{B} = \boldsymbol{0}$, Eq. (1.2) reduces into

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\boldsymbol{E}. \tag{1.4}$$

If the electric field is uniform, it holds $\boldsymbol{E} = \boldsymbol{const}.$, so that we can write

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q\boldsymbol{E}}{m} \quad \Rightarrow \quad \mathbf{v} = \int \frac{q\boldsymbol{E}}{m}\mathrm{d}t = \frac{q\boldsymbol{E}}{m}t + \boldsymbol{C}_1,$$

where the constant of integration[1] can be determined from the initial condition (1.3) as $\boldsymbol{C}_1 = \mathbf{v}_0$, so that for the particle velocity we get

$$\mathbf{v} = \frac{q\boldsymbol{E}}{m}t + \mathbf{v}_0. \tag{1.5}$$

The speed of the particle changes with the time, if the vectors $\boldsymbol{E}$ and $\mathbf{v}_0$ are parallel, the motion takes place along a straight line, otherwise the trajectory is curved.

Integration of Eq. (1.5) results in the time dependence of the particle position vector as

$$\boldsymbol{r} = \int \mathbf{v}\mathrm{d}t = \frac{1}{2}\frac{q\boldsymbol{E}}{m}t^2 + \mathbf{v}_0 t + \boldsymbol{C}_2,$$

where the constant of integration can be determined employing the initial condition (1.3) as $\boldsymbol{C}_2 = \boldsymbol{r}_0$, namely,

$$\boldsymbol{r} = \frac{1}{2}\frac{q\boldsymbol{E}}{m}t^2 + \mathbf{v}_0 t + \boldsymbol{r}_0. \tag{1.6}$$

1.2.3 Motion of a charged particle in a uniform magnetic field

If it holds $\boldsymbol{E} = \boldsymbol{0}$, the equation of motion (1.2) can be written in form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m}\mathbf{v} \times \boldsymbol{B}. \tag{1.7}$$

If the magnetic field is uniform, i.e., $\boldsymbol{B} = \boldsymbol{const}.$, the coordinate system can be, without the loss of generality, oriented such that the coordinate z has the direction of the magnetic field vector, namely, $\boldsymbol{B} = (0, 0, B)$, where $B > 0$. Then the cross-product in Eq. (1.7) reads

$$\mathbf{v} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = v_y B\boldsymbol{i} - v_x B\boldsymbol{j},$$

so that Eqs. (1.7) can be written as

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{qB}{m}v_y, \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{qB}{m}v_x, \qquad \frac{\mathrm{d}v_z}{\mathrm{d}t} = 0. \tag{1.8}$$

The third of Eqs. (1.8) is decoupled from the first two and it follows from it that the z-component of the velocity vector (in the direction of the magnetic field) is time-independent and it holds

$$v_z = v_{z0}, \qquad z = v_{z0}t + z_0. \tag{1.9}$$

The solution of the set of the first two of Eqs. (1.8) can be most easily found employing the following trick. We introduce a complex velocity $\hat{v} \equiv v_x + \mathrm{j}v_y$, multiply the second of Eqs. (1.8) with the imaginary unit and add to the first of the equations. This results in

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} + \mathrm{j}\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = \frac{qB}{m}\left(v_y - \mathrm{j}v_x\right) = -\mathrm{j}\frac{qB}{m}\hat{v} \qquad \Rightarrow \qquad \frac{\mathrm{d}\hat{v}}{\mathrm{d}t} + \mathrm{j}\omega_\mathrm{c}\hat{v} = 0, \tag{1.10}$$

where

$$\omega_\mathrm{c} = \frac{qB}{m} \tag{1.11}$$

[1] More precisely said, it is a constant vector.

is the so-called cyclotron frequency. It can be easily proved by direct substitution, that the solution of Eq. (1.10) reads

$$\hat{v} = \hat{C}\mathrm{e}^{-\mathrm{j}\omega_\mathrm{c}t},$$

where $\hat{C}$ is a constant of integration, which can be found by substituting the initial condition $\hat{v}(t=0) = v_{x_0} + \mathrm{j}v_{y0}$ into the previous result; it applies $\hat{C} = v_{x_0} + \mathrm{j}v_{y0}$, so that we can write

$$\hat{v} = v_x + \mathrm{j}v_y = (v_{x0} + \mathrm{j}v_{y0})\left(\cos\omega_\mathrm{c}t - \mathrm{j}\sin\omega_\mathrm{c}t\right).$$

The comparison of the real and imaginary parts on the LHS and RHS of the previous relation yields in

$$\begin{aligned} v_x &= v_{x0}\cos\omega_\mathrm{c}t + v_{y0}\sin\omega_\mathrm{c}t, \\ v_y &= v_{y0}\cos\omega_\mathrm{c}t - v_{x0}\sin\omega_\mathrm{c}t. \end{aligned}$$

If we introduce the magnitude of the component of the initial velocity perpendicular to the magnetic field as

$$v_{\perp 0} \equiv \sqrt{v_{x0}^2 + v_{y0}^2},$$

the previous results can be formally expressed as

$$v_x = v_{\perp 0}\left(\frac{v_{x0}}{v_{\perp 0}}\cos\omega_\mathrm{c}t + \frac{v_{y0}}{v_{\perp 0}}\sin\omega_\mathrm{c}t\right) = v_{\perp 0}\left(\cos\omega_\mathrm{c}t\cos\delta + \sin\omega_\mathrm{c}t\sin\delta\right). \tag{1.12a}$$

$$v_y = v_{\perp 0}\left(\frac{v_{y0}}{v_{\perp 0}}\cos\omega_\mathrm{c}t - \frac{v_{x0}}{v_{\perp 0}}\sin\omega_\mathrm{c}t\right) = v_{\perp 0}\left(\cos\omega_\mathrm{c}t\sin\delta - \sin\omega_\mathrm{c}t\cos\delta\right). \tag{1.12b}$$

where we have introduced

$$\cos\delta \equiv \frac{v_{x0}}{v_{\perp 0}}, \qquad \sin\delta \equiv \frac{v_{y0}}{v_{\perp 0}} \qquad \Rightarrow \qquad \tan\delta \equiv \frac{v_{y0}}{v_{x0}}.$$

Employing the well known formulas for the summation of trigonometric functions, Eqs. (1.12) can be rewritten as

$$v_x = v_{\perp 0}\cos(\omega_\mathrm{c}t - \delta), \qquad v_y = -v_{\perp 0}\sin(\omega_\mathrm{c}t - \delta). \tag{1.13}$$

For the speed of the particle it directly follows from Eqs. (1.9) and (1.13) that

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{v_{\perp 0}^2 + v_{z0}^2} = \sqrt{v_{x0}^2 + v_{y0}^2 + v_{z0}^2} = v_0 = const.,$$

the particle speed in a uniform magnetic field is constant; however, its direction does change in time. The integration of Eqs. (1.13) leads to the x- and y-component of the particle position vector

$$\begin{aligned} x &= \int v_x \mathrm{d}t = \frac{v_{\perp 0}}{\omega_\mathrm{c}}\sin(\omega_\mathrm{c}t - \delta) + C_x, \\ y &= \int v_y \mathrm{d}t = \frac{v_{\perp 0}}{\omega_\mathrm{c}}\cos(\omega_\mathrm{c}t - \delta) + C_y. \end{aligned}$$

If we introduce a new quantity

$$R_\mathrm{c} \equiv \frac{v_{\perp 0}}{\omega_\mathrm{c}} = \frac{mv_{\perp 0}}{qB}, \tag{1.14}$$

after the determination of the constants of integration, the trajectory of the charged particle in a uniform magnetic field can be expressed as

$$x = R_\mathrm{c}\sin(\omega_\mathrm{c}t - \delta) + R_\mathrm{c}\sin\delta + x_0, \tag{1.15a}$$

$$y = R_\mathrm{c}\cos(\omega_\mathrm{c}t - \delta) - R_\mathrm{c}\cos\delta + y_0. \tag{1.15b}$$

$$z = v_{z0}t + z_0. \tag{1.15c}$$

To sum up, it follows from Eqs. (1.15) that the trajectory of a charged particle in a uniform magnetic field is a helix with the radius $|R_\mathrm{c}|$ (the cyclotron radius), oriented along the magnetic field direction.

If the charged particle enters the magnetic field in the perpendicular direction and thus it applies $v_{z0} = 0$, it moves further along a circular trajectory with the radius $|R_\mathrm{c}|$ and the period $T_\mathrm{c} = 2\pi/\omega_\mathrm{c}$.

1.3 Experiment

1.3.1 Principle

The specific charge of the electron can be determined as follows, see Fig. 1.1. Electrons are emitted from the heated electrode (cathode) of an electron gun, and then accelerated towards the positive electrode. Assume that there is a uniform electric field between the planar electrodes with a mutual distance h, for the magnitude of whose intensity it applies $E = U/h$, where U is the voltage between the electrodes. If the initial velocity of an emitted electron is small, it will move in the electric field along a straight line against the direction of the electric field. For the speed and the distance traveled on time, see the relations (1.5) and (1.6), it applies

$$v = \frac{eE}{m_\mathrm{e}}t, \qquad s = \frac{1}{2}\frac{eE}{m_\mathrm{e}}t^2,$$

where e is the elementary charge, and m_e is the electron mass. The electron thus reaches the anode (after having passed the distance h) at the time

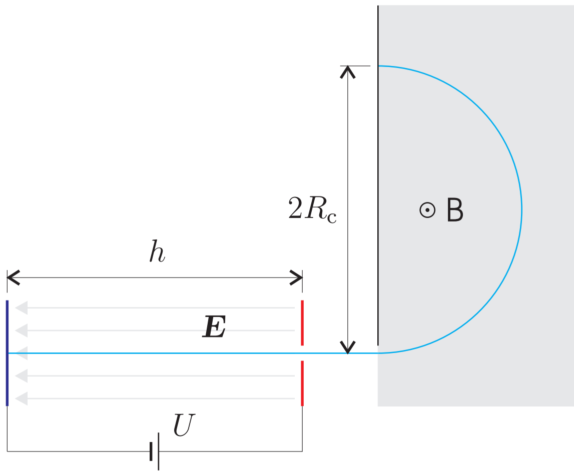


Figure 1.1: Schematics of the experimental set-up.

$$t_h = \sqrt{\frac{2hm_\mathrm{e}}{eE}},$$

and for its speed it holds

$$v = \frac{eE}{m_\mathrm{e}}t_h = \sqrt{\frac{2ehE}{m_\mathrm{e}}} = \sqrt{\frac{2eU}{m_\mathrm{e}}}.$$

After the electron passes through the hole in the anode, it moves along a straight line with a constant speed (the electric field is concentrated between the electrodes only) until it flies *perpendicularly* into a uniform magnetic field with magnitude B. Here, it moves along a part of a circular trajectory for whose (cyclotron) radius, see the relation (1.14), it applies

$$R_\mathrm{c} = \frac{m_\mathrm{e}v}{eB} = \sqrt{\frac{2m_\mathrm{e}U}{eB^2}}.$$

From here, by the measurement of the (cyclotron) trajectory radius, we can determine the electron specific charge e/m_e as

$$\frac{e}{m_\mathrm{e}} = \frac{2U}{B^2R_\mathrm{c}^2}. \tag{1.16}$$

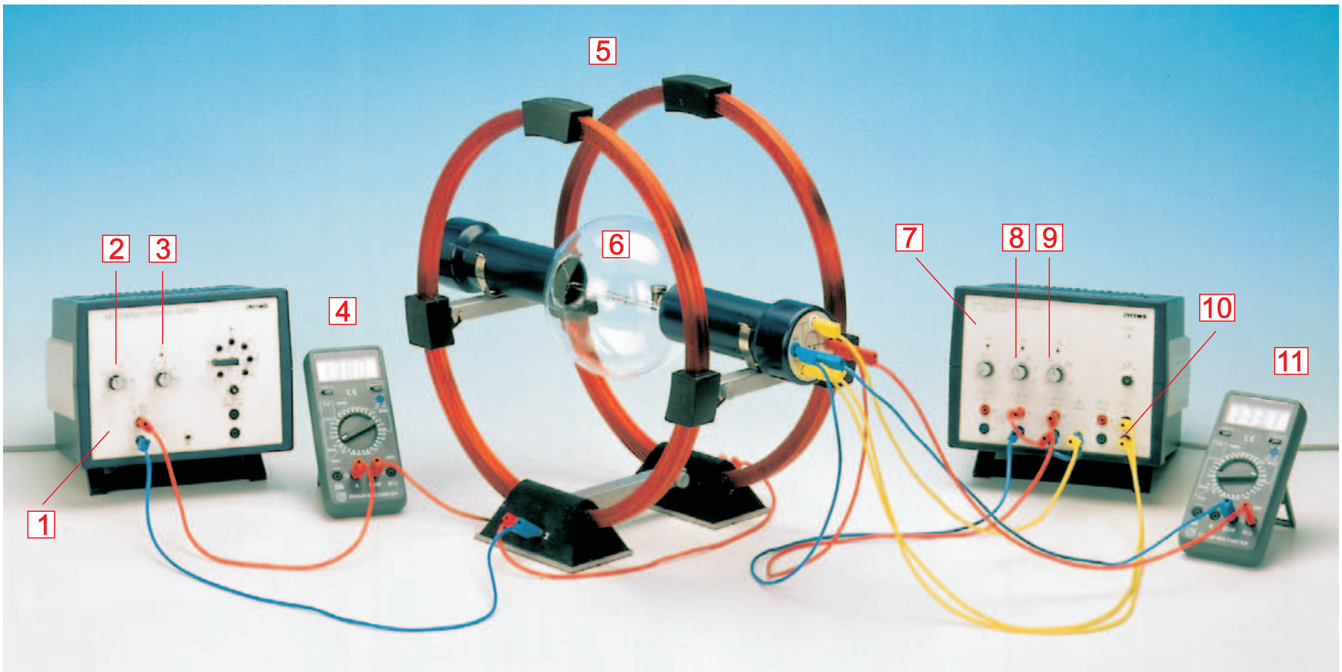


Figure 1.2: Experimental set-up: $\boxed{1}$ – power supply for the Helmholtz coils, $\boxed{2}$ – voltage adjustment, $\boxed{3}$ – current limiter, $\boxed{4}$ – ammeter for the measurement of the current through Helmholtz coils, $\boxed{5}$ – pair of coils in the Helmholtz arrangement, $\boxed{6}$ – argon-filled narrow-beam tube, $\boxed{7}$ – low-voltage power supply for the electron gun, $\boxed{8}$ – adjustment of the grid voltage 0–50 V, $\boxed{9}$ – adjustment of the anode voltage 0–300 V, $\boxed{10}$ – output 6.3 V∼ for the cathode heating, $\boxed{11}$ – voltmeter for the measurement of the accelerating voltage.

1.3.2 Experimental set-up

The experimental set-up for the determination of the specific charge of the electron is shown in Fig. 1.2. The electron beam is emitted by an electron gun $\boxed{6}$ in a tube filled with argon (pressure about 0.1 Pa). When the accelerated electrons collide with argon atoms, they are ionized, and when the resulting ions recombine into neutral atoms, photons are emitted, so that a narrow electron beam can be observed in the tube.

The speed of the electrons can be adjusted via the accelerating voltage U, which is the sum of the grid voltage (set in the range 0–50 V by the potentiometer $\boxed{8}$), and the anode voltage (set in the range 0–250 V by the potentiometer $\boxed{9}$), see Fig. 1.3. The cathode of the electron gun is heated by the AC voltage 6.3 V.

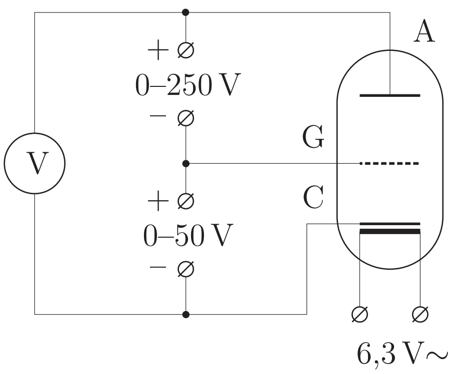


Figure 1.3: Wiring diagram of the narrow-beam tube–the electron gun.

The magnetic field in which the electron beam is allowed to curve is generated in the axis of a pair of coils in Helmholtz arrangement (Fig. 1.2, $\boxed{5}$). These are two identical coaxial coils through which the same current passes in the same direction. It can be shown, see the Appendix, that if the mutual distance of the coils is equal to their radius, the magnetic field vector in the axis of the coils is approximately constant and its magnitude is

$$B \approx B_0 = \frac{8}{5\sqrt{5}} \frac{\mu_0 N I}{a}, \tag{1.17}$$

where $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{N{\cdot}A^{-2}}$ is the magnetic constant, N is the number of turns of each coil (in this case, $N = 154$), I is the current through the coils, and a is their radius (in this case, $a = 200\,\mathrm{mm}$). The Helmholtz coils work correctly only if current flows through them in the same direction (otherwise the magnetic field at their center is zero). The current through the Helmholtz coils is adjusted by a current limiter $\boxed{3}$ on the small-voltage power supply $\boxed{1}$, which prevents it from dropping when the coils heat up. The output voltage should be set to the maximum value using the potentiometer $\boxed{2}$.

If the accelerated electrons enter the magnetic field perpendicularly, they move along circular trajectories that can be observed in the tube. If the trajectory is helical, the bulb must be rotated along its axis so that the trajectories are circular. The radii of the trajectories are not measured but adjusted. In the tube, at distances $l = 4$, 6, 8 and 10 cm from the electron gun, luminous traces are placed; if the electron beam hits a given trace, it lights up and the cyclotron radius (radius of the circular trajectory) **is equal to half the distance** l.

1.3.3 Safety during the measurement

The accelerating voltage of the electron gun can be as high as 300 V. For this reason, do not disconnect or tamper with the tube power supply circuit. **Ask the instructor to turn the experiment on and off**.

1.3.4 Procedure

1. Before switching on the power supply of the electron gun $\boxed{7}$, the potentiometers $\boxed{8}$ and $\boxed{9}$ must be set to the minimum (zero) value.

2. Ask the instructor to switch the experiment on.

3. After switching on the power supply $\boxed{7}$ you need to let the cathode of the electron gun glow for about 2 minutes before you start increasing the accelerating voltage. This extends the lifetime of the electron gun cathode.

4. For various accelerating voltages U (the experiment works well for voltages greater than about 100 V), find the currents through the Helmholtz coils (and hence the magnetic field) when the electrons hit the luminous traces, i.e., when the cyclotron radius of their trajectories can be determined. Make the measurements at least sixteen times.

5. For each combination of the set and measured values, calculate the specific charge of the electron using the formula (1.16). From the calculated values, determine the mean value and the uncertainty of the measurement.

6. When you finish the measurement, set the anode and grid voltage source potentiometers to minimum – this extends the lifetime of the electron-gun cathode. Ask the instructor to switch the experiment off.

1.4 References

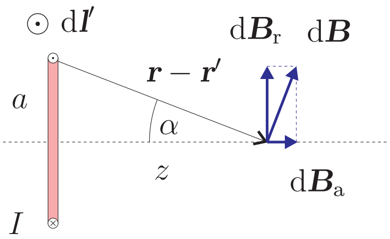
1. B. Sedlák, I. Štoll: *Elektřina a magnetismus*, Academia, Praha, 2002.

2. David J. Griffiths, *Introduction to Electrodynamics*, Prentice Hall, New Yersey, 1999.

1.5 Appendix

1.5.1 Magnetic field in the axis of a circular loop

We use the Biot-Savart-Laplace law to calculate the magnetic field in the axis of a circular loop. Let the loop have radius a and current I flows through it.



An element of the current-carrying loop $\mathrm{d}\boldsymbol{l}'$ generates on its axis magnetic field

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{l}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

where $\mu_0 = 4\pi\cdot 10^{-7}\,\mathrm{N{\cdot}A^{-2}}$ is the magnetic constant. Thanks to the fact that the individual vectors are perpendicular, it holds for the magnetic field element magnitude

$$\mathrm{d}B = \frac{\mu_0 I \mathrm{d}l'}{4\pi(a^2+z^2)}.$$

Its component in the axis direction is

$$\mathrm{d}B_z = \mathrm{d}B\sin\alpha = \frac{\mu_0 I a \mathrm{d}l'}{4\pi(a^2+z^2)^{3/2}}.$$

As the component $\mathrm{d}\boldsymbol{B}_z$ is the same for all the loop elements we get for the total magnetic field in the loop axis

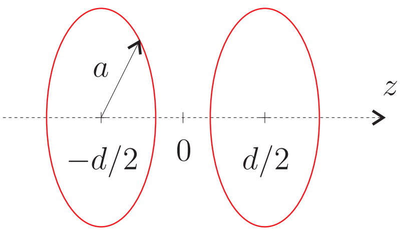
$$B_z = \oint_{\mathcal{L}} \frac{\mu_0 I a \mathrm{d}l'}{4\pi(a^2+z^2)^{3/2}} = \frac{\mu_0 I a}{4\pi(a^2+z^2)^{3/2}}\oint_{\mathcal{L}} \mathrm{d}l' = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}.$$

As the radial component of the magnetic field vector on the loop axis is zero due to the symmetry, the magnetic field vector on the axis has the axial direction and for its magnitude it applies $B = B_z$.

If the loop had N turns, the total current passing through it would be NI, so that employing the superposition principle (the magnetic field is a linear function of current) results in the total magnetic field on the axis of this loop

$$B = \frac{\mu_0 N I a^2}{2(a^2+z^2)^{3/2}}. \tag{1.18}$$

1.5.2 Coils in Helmholtz arrangement



Helmholtz coils consist of two identical coaxial circular loops of radius a, each with N turns, through which the same current I flows in the same direction. If these loops are placed at a mutual distance $d = a$, the magnetic field in the axis between the loops is approximately uniform. We will prove this statement in the following text.

We place the center of the z-axis (symmetry axis) at the center between the coils, see the figure. According to Eq. (1.18), magnetic field on the axis is given as

$$B = \frac{\mu_0 N I a^2}{2}\left\{\frac{1}{[a^2+(z-d/2)^2]^{3/2}} + \frac{1}{[a^2+(z+d/2)^2]^{3/2}}\right\}.$$

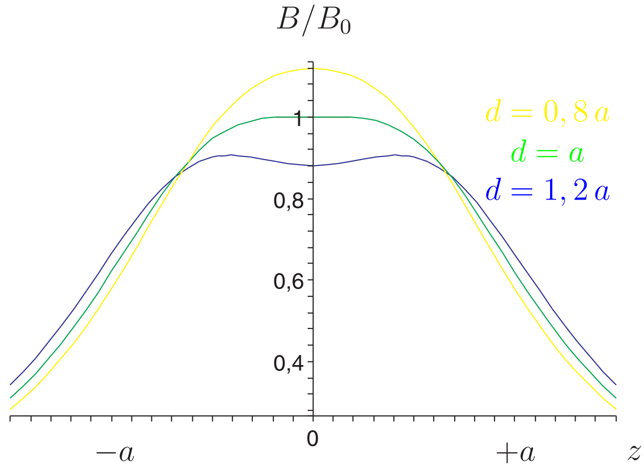


Figure 1.4: Magnetic field in the axis of Helmholtz coils for different distances d.

It is self-evident that the less dependent the field between the loops is on the z-coordinate (the more uniform it is), the smaller the derivatives of the magnetic field with respect to the z-coordinate will be. For the first derivative it applies

$$\frac{\mathrm{d}B}{\mathrm{d}z} = -\frac{3\mu_0 N I a^2}{2}\left\{\frac{z - d/2}{[a^2 + (z - d/2)^2]^{5/2}} + \frac{z + d/2}{[a^2 + (z + d/2)^2]^{5/2}}\right\},$$

Obviously, $\mathrm{d}B/\mathrm{d}z = 0$ for $z = 0$ (symmetry). For the second derivative it applies

$$\frac{\mathrm{d}^2B}{\mathrm{d}z^2} = -\frac{3\mu_0 N I a^2}{2}\left\{\frac{a^2 - 4(z - d/2)^2}{[a^2 + (z - d/2)^2]^{7/2}} + \frac{a^2 - 4(z + d/2)^2}{[a^2 + (z + d/2)^2]^{7/2}}\right\}.$$

For $z = 0$ it applies

$$\left.\frac{\mathrm{d}^2B}{\mathrm{d}z^2}\right|_{z=0} = 3\mu_0 N I a^2 \frac{d^2 - a^2}{(a^2 + d^2/4)^{7/2}}.$$

The second derivative is therefore zero if the distance between the loops is equal to their radius. For the magnitude of the magnetic field vector in the axis between the loops it then applies

$$B \approx B_0 = \frac{8}{5\sqrt{5}}\frac{\mu_0 N I}{a}. \tag{1.19}$$

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