Laboratory experiment

Measurement of speed of sound employing sonar

1.1 Task

Determine the speed of sound in air by measuring the time between transmission and registration of reflected ultrasonic pulses. Compare the measured value of the speed of sound with the calculated value.

1.2 Linearized equations of acoustic field

At the beginning, we derive the basic relationships with which we describe acoustic field and show that a wave equation can be derived from them. For simplicity, we will consider only a one-dimensional case.

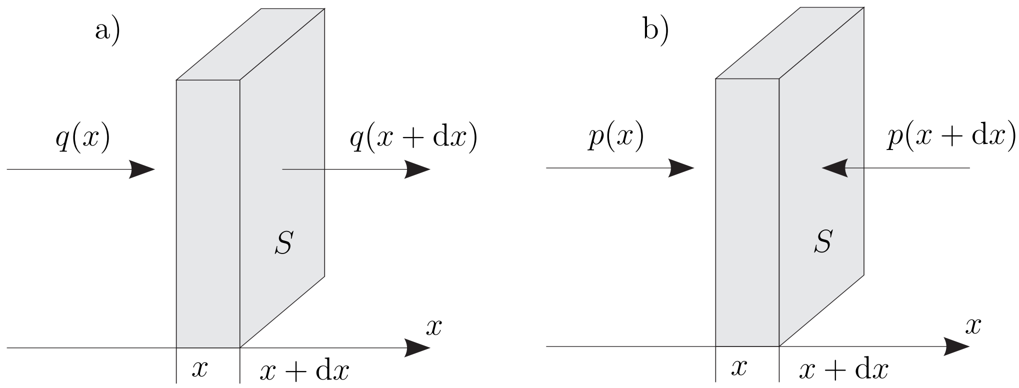

Figure 1.1: Regarding the continuity and momentum equation.

1.2.1 Continuity equation

We derive the one-dimensional continuity equation for a fluid moving along the x axis, see Fig. 1.1a).

Assume a fluid with mass flux density $q(x)$ flowing from the left side into an elementary control volume $\mathrm{d}V$. The fluid with the flux density $q(x + \mathrm{d}x)$ flows out of the control volume at the right side. The increase of the mass of $\mathrm{d}V$ per unit time is then

$$\frac{\mathrm{d}m}{\mathrm{d}t} = S\left[q(x) - q(x + \mathrm{d}x)\right], \tag{1.1}$$

where m is the mass of the elementary volume and S is the cross-sectional area. As the flux density changes only a little within the elementary distance $\mathrm{d}x$, we can use the first two terms of the Taylor series to get

$$q(x + \mathrm{d}x) = q(x) + \frac{\partial q(x)}{\partial x}\mathrm{d}x. \tag{1.2}$$

As the mass flux density $q = \rho v$, where ρ is the mass density, and v is the fluid velocity, and as the mass of the volume element $m = \rho S \mathrm{d}x$, substitution of Eq. (1.2) into Eq. (1.1) results in the one-dimensional continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x}. \tag{1.3}$$

The total fluid density ρ has two components. There is an ambient fluid density ρ_0, to which there is superimposed so-called acoustic density ρ', which is related to perturbations propagating in in fluid. Thus, it holds $\rho = \rho_0 + \rho'$, and $\rho' \ll \rho_0$. Considering this fact and assuming that the ambient fluid density is constant, we can neglect the terms $\partial\rho_0/\partial t$ and $\partial(\rho' v)/\partial x$, in the continuity equation and thus we linearize it as

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \frac{\partial v}{\partial x}. \tag{1.4}$$

1.2.2 Momentum equation

We derive the one-dimensional equation of motion for a perfect (non-viscous) fluid, see Fig. 1.1b).

There is the fluid pressure $p(x)$ acting on the elementary volume $\mathrm{d}V$ from the left side and pressure $p(x + \mathrm{d}x)$ acting from the right side. If we neglect the volume forces (gravity, inertial force) we can write the momentum equation in form

$$\mathrm{d}m \frac{\partial v}{\partial t} = S \left[p(x) - p(x + \mathrm{d}x)\right], \tag{1.5}$$

where $\mathrm{d}m$ is the mass of the elementary control volume $\mathrm{d}V$.

As the fluid pressure changes only a little within the elementary distance $\mathrm{d}x$, we can use the first two terms of the Taylor series to get

$$p(x + \mathrm{d}x) = p(x) + \frac{\partial p(x)}{\partial x}\mathrm{d}x. \tag{1.6}$$

Substituting Eq. (1.6) into Eq. (1.5) and using $\mathrm{d}m = \rho S \mathrm{d}x$ results in the momentum equation in form

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}. \tag{1.7}$$

The total pressure p has two components. There is the acoustic pressure p', connected with the perturbations propagating in the fluid, superimposed on the ambient (barometric) pressure p_0, so that $p = p_0 + p'$. The ambient pressure can be considered constant in space and time. As it also holds $\rho' \ll \rho_0$ (which we have used in the previous paragraph), the momentum equation can be linearized and written as

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial x}. \tag{1.8}$$

1.2.3 Equation of state

When an acoustic wave propagates in a gas, the gas condenses and dilutes very quickly and it also conducts the heat quite poorly. For this reason, we can assume that the heat is not exchanged and thus the propagation of sound can be considered an adiabatic process, which can be described by the adiabatic equation of state.

$$pV^{\gamma} = p_0 V_0^{\gamma} \qquad \Longrightarrow \qquad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{1.9}$$

where V is the volume (of an acoustic particle) and γ is the adiabatic exponent. If we, again, only consider weak perturbations propagating in the fluid, Eq. (1.9) can be linearized. Taylor series of Eq. (1.9) calculated for ρ_0, discarding the higher-order terms, results in

$$p - p_0 = \frac{\gamma p_0}{\rho_0}(\rho - \rho_0), \tag{1.10}$$

which can be rewritten as

$$p' = c_0^2 \rho', \tag{1.11}$$

where

$$c_0^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} = \frac{\gamma p_0}{\rho_0}, \tag{1.12}$$

as it will be shown in the following text, is the square of the speed of sound.

The above formula can be rewritten employing the equation of state of an ideal gas as follows

$$pV = nRT \quad \rightarrow \quad p\frac{m}{\rho} = nRT \quad \rightarrow \quad p = \frac{n}{m}\rho RT \quad \rightarrow \quad p = \frac{\rho RT}{M},$$

where R is the molar gas constant, T is the temperature (in kelvins), n is the amount of substance, and M is the molar mass of the respective gas. Substitution of this formula into Eq. (1.12) results in

$$c_0 = \sqrt{\frac{\gamma R}{M}T}. \tag{1.13}$$

From here, it follows that for a given temperature, the speed of sound is greater in gases with a lower molar mass and that for a given gas the speed of sound is proportional to the square root of the thermodynamic temperature. For temperature in degrees Celsius θ, employing relation $T = T_0 + \theta$ ($T_0 = 273.15$ K) relation (1.13) gets the form

$$c_0 = \sqrt{\frac{\gamma R T_0}{M}\left(1 + \frac{\theta}{T_0}\right)}. \tag{1.14}$$

For $\theta \ll T_0$ the formula can be linearized employing the first two terms of the Taylor series as

$$c_0 \approx \sqrt{\frac{\gamma R T_0}{M}} + \frac{1}{2}\sqrt{\frac{\gamma R}{M T_0}}t \tag{1.15}$$

and substituting the corresponding values for air ($\gamma = 7/5$, $R = 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$, $\mathrm{M} = 28.96\,\mathrm{g\,mol^{-1}}$) we retrieve the well known formula

$$c_0 \approx 331.06 + 0.61\theta \qquad [\mathrm{m/s},\ ^{\circ}\mathrm{C}]. \tag{1.16}$$

1.2.4 Wave equation

In this section we prove that perturbations in fluids propagate as waves. The one-dimensional linearized equations (1.4), (1.8), (1.11) form a set of partial differential equations describing acoustic field in an inviscid, homogeneous and quiescent fluid

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \frac{\partial v}{\partial x}, \tag{1.17a}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial x}, \tag{1.17b}$$

$$p' = c_0^2 \rho'. \tag{1.17c}$$

The set of equations can be rewritten as follows by eliminating the acoustic density and velocity. We take the time derivative of Eq. (1.17a) and we take the derivative of Eq. (1.17b) with respect to the spatial coordinate so that we get

$$\frac{\partial^2 \rho'}{\partial t^2} = -\rho_0 \frac{\partial^2 v}{\partial x \partial t}, \tag{1.18a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t \partial x} = -\frac{\partial^2 p'}{\partial x^2}. \tag{1.18b}$$

From here we eliminate the second order mixed derivatives and we substitute for the acoustic density from Eq. (1.17c). This way we obtain the wave equation for the acoustic pressure in the form

$$\frac{\partial^2 p'}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0. \tag{1.19}$$

We can easily verify by direct substitution (taking the corresponding derivatives) that the solution to the wave equation (1.19) can be written as

$$p'(x, t) = f(x - c_0 t) + g(x + c_0 t), \tag{1.20}$$

where the function f represents a wave propagating in the positive x direction with phase velocity c_0, and the function g represents a wave propagating with the same speed in the negative x direction. The shape of the functions f and g is related to the initial and boundary conditions for a given problem.

1.3 Measurement of speed of sound

The set-up for the measurement of the speed of sound is very simple, it works on principle of sonar and it is schematically depicted in Fig. 1.2.

A short ultrasonic pulse is sent from the ultrasound transmitter, it propagates in the air as a sound wave and after being reflected from the screen it is received by an ultrasonic receiver. We can calculate the speed of sound c_0 from the time between the transmission and registration the reflected ultrasonic pulse and the distance l it has traveled (see Fig. 1.2). For different distances l_i we obtain different propagation times Δt_i. As there is the well known formula for the motion with constant velocity $s = vt + s_0$, where s, s_0 are the distances and v is the velocity, we can calculate the speed of sound such that, employing the least squares method, we approximate the measured values Δt_i, l_i by a straight line (linear polynomial) $l = A\Delta t + B$, where we determine the speed of sound as $c_0 = A$.

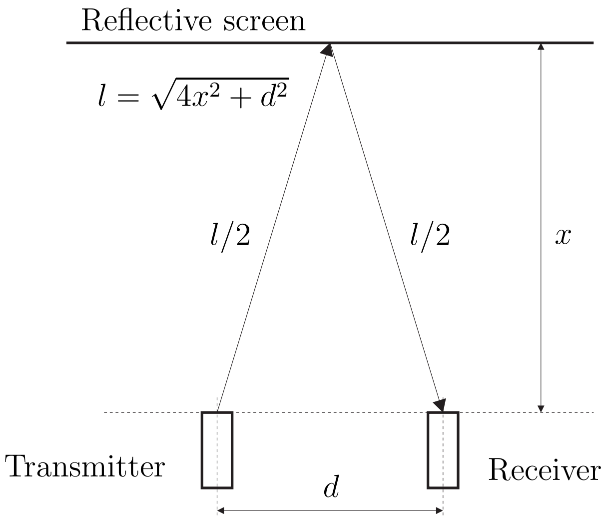

Figure 1.2: Experimental set-up.

1.4 Instructions

1.4.1 Procedure

1. Check the experimental set-up.

2. Measure the time interval between the transmission and registration of ultrasonic pulses for at least 10 different distances between the reflecting screen and the transmitter-receiver set.

3. Use the least squares method to calculate the speed of sound in the air. Compare this value with the value calculated for a given temperature using the formula (1.16).

   You can use the script *An universal tool for plotting graphs - least squares method* at website *http:\\planck.fel.cvut.cz/praktikum/* to create a graph, calculate the speed of sound and its uncertainty.

1.4.2 Experimental set-up

It is not necessary to disconnect cables and individual devices after the measurement, so do not do so unnecessarily. If some cables are disconnected (or something does not work), the connection procedure is given below.

Plug the ultrasonic (USC) transmitter into connector TR1 of the USC unit (10, Fig. 1.3) and employing the button 4, switch the USC unit into the burst regime (Burst). Plug the USC receiver into the input BNC connector of the USC unit (14, Fig. 1.3). Adjust the output signal amplitude (potentiometer 6) and/or the input gain of the USC unit (switch 1 and potentiometer 2) so that the amplifier of the USC unit is not overloaded. The overload is indicated by LED OVL (3, Fig. 1.3).

Connect the synchronization output of the USC unit (BNC connector 11) to input 1 of the oscilloscope. Connect the analog output of the USC unit (BNC connector 13) with the input 2 of the oscilloscope.

1.4.3 Digital oscilloscope Agilent DSO-X 2012A: setting and operation

- Switch on the oscilloscope using the button (1, Fig. 1.4) at the left-bottom side of the front panel.

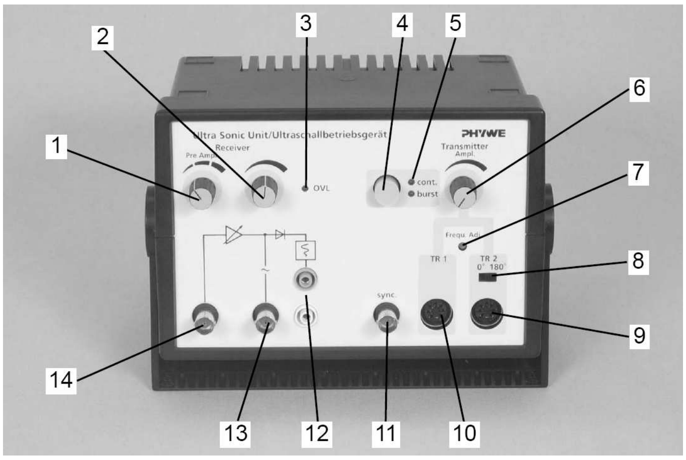

Figure 1.3: Ultrasonic (USC) unit. 1 - Rotary switch for input signal amplification selection; 2 - Potentiometer for the input signal amplification adjustment; 3 - Overload LED indicator (OVERLOADED); 4, 5 - Operation mode selection button with LED indicator: cont. indicates the continuous mode, burst indicates the burst mode; 6 - Potentiometer for the output signal amplitude adjustment; 7 - USC frequency adjustment; 8 - Switch for the phase reversal of the output USC signal; 9, 10 - Terminals for the connection of USC transmitters; 11 - Analog power output; 12 - Output of amplified and rectified signal of the USC receiver; 13 - Output of amplified signal of the USC receiver; 14 - Input for the connection of USC receiver.

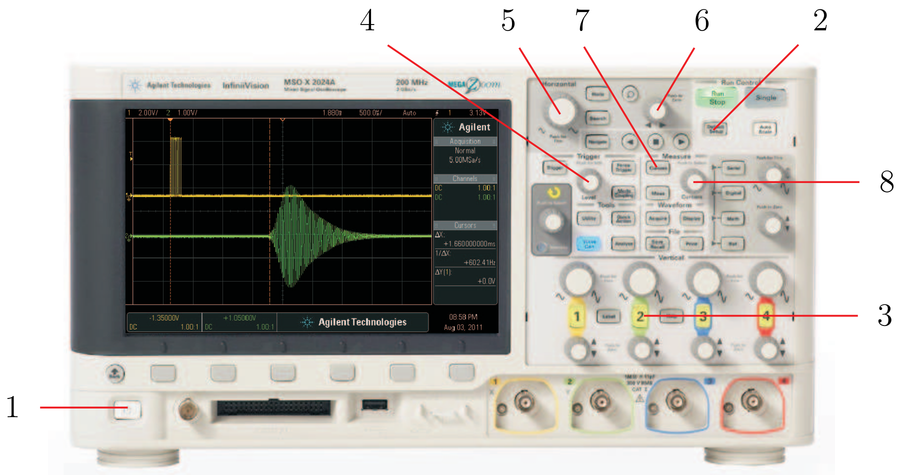

Figure 1.4: Digital oscilloscope Agilent DSO-X 2012A.

- Push the button *Default setup* (2, Fig. 1.4) to reset the previous oscilloscope setting.
- Push the button *2* (3, Fig. 1.4), to activate the second input channel (the button will light on).
- Rotate the knob *Level* in section *Trigger* (4, Fig. 1.4) to set the synchronization input signal level of the transmitted signal. The displayed signal on the oscilloscope should get stationary.
- Use knob *Horizontal* in section *Horizontal* (5, Fig. 1.4) and the knob with the right and left arrows (6, Fig. 1.4) to adjust the display, so that you can see the transmitted and received pulse well. Use the rotary knobs in section *Vertical* to adjust the scale of the displayed pulses.
- Push the button *Cursors* situated in section *Measure* (7, Fig. 1.4) to activate cursors (you

should see a vertical line on the display at the beginning of the synchronization pulse).

- Push the rotary knob *Cursors* in section *Measure* (8, Fig. 1.4) to display the menu in which you can select (by rotating this knob) required cursor (*X1*, *X2*), make the selection by pushing this knob. Adjust the position of the cursors by rotating this knob (to the beginning of the pulses). After setting the position of both the cursors, read the time separation ΔX between them on the oscilloscope display.

1.5 References

1. Zdeněk Škvor: Akustika a elektroakustika, *Academia*, Praha, 2001

2. Jiří Bajer: Mechanika 3, *Univerzita Palackého v Olomouci*, Olomouc, 2006.

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