Laboratory experiment

Stefan-Boltzmann's law of radiation

1.1 Task

Check the validity of the Stefan-Boltzmann's law by measuring the temperature dependence of the power emitted by the filament of a light bulb.

1.2 Theory

1.2.1 Black body

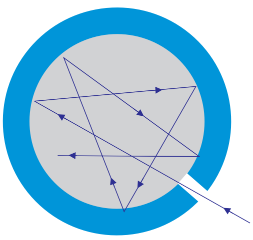


Figure 1.1: Model of a black body

It is well-known from thermodynamics that there are three ways of transferring thermal energy from point to point: conduction, convection, and radiation. The first two mechanisms require a material environment to exist, whereas the transfer of thermal energy by radiation can take place in vacuum. By thermal radiation is generally meant all radiation emitted by the surface of a body with a non-zero absolute temperature. The spectrum of this radiation is continuous.

When thermal radiation strikes the opaque surface of a body, some of it is reflected back, and some is absorbed. By definition, a black body is a body whose surface absorbs thermal radiation perfectly and therefore reflects nothing. A rough black surface is close to a black body, but a more perfect prototype of the black body is more like a cavity with a small hole. Due to the finite reflectivity of the inner walls, almost every ray that enters the cavity is quickly absorbed. The hole thus appears black, and is a good approximation of the black body. Although a black body absorbs all incident radiation, as long as it is at a non-zero (thermodynamic) temperature, it emits radiation itself.

In 1859, Gustav Robert Kirchhoff investigated the laws of thermal radiation. Using the general laws of thermodynamics, he showed that the greater the surface absorption of a body, the better the thermal radiation it emits, so that the best emitter of thermal radiation is paradoxically the black body. He also showed that the intensity of the radiation of the black body is only a function of temperature $M^{\mathrm{e}} = f(T)$.

The spectrum of black-body radiation is governed by Planck's radiation law[1], which can be expressed, for example, in the following form

$$M^{\mathrm{e}}_{\lambda}(T, \lambda)\mathrm{d}\lambda = \frac{2\pi hc^2}{\lambda^5(\mathrm{e}^{hc/\lambda k_{\mathrm{B}}T} - 1)}\mathrm{d}\lambda, \tag{1.1}$$

[1] This formula was derived in 1900 by German physicist Max Planck and it laid the foundations of quantum mechanics.

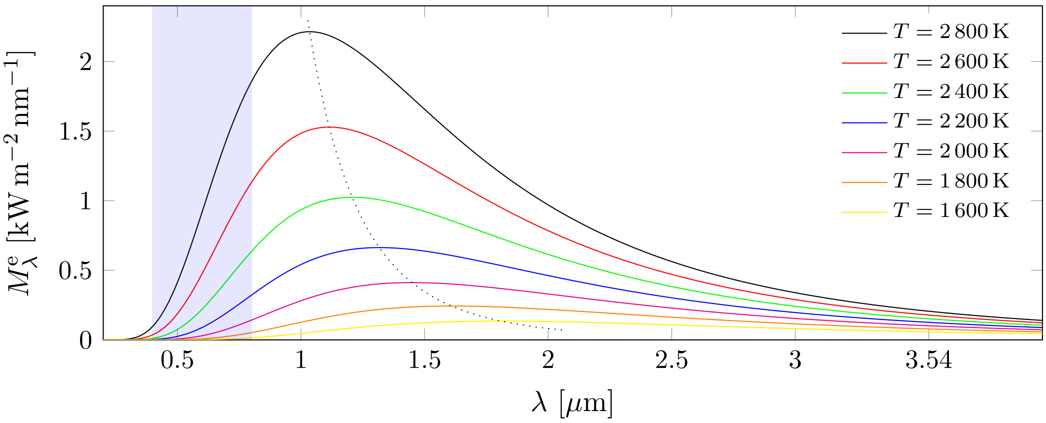


Figure 1.2: Spectrum of black-body radiation for different temperatures. The blue rectangle delimits the visible part of the spectrum.

where $M^{\mathrm{e}}_{\lambda}(T, \lambda)$ is the so-called spectral intensity of radiation, λ is the wavelength, T is the thermodynamic temperature (in kelvins), h is the Planck constant, c is the speed of light in vacuum, and k_{B} is the Boltzmann constant. The expression $M^{\mathrm{e}}_{\lambda}(T, \lambda)\mathrm{d}\lambda$ represents the power radiated by a black body into all directions trough one meter squared of its surface in the wavelength-band $(\lambda, \lambda + \mathrm{d}\lambda)$. It can be seen from Eq. (1.1) that this radiated power only depends on the temperature T. The spectral intensity of black-body radiation for individual temperatures is depicted in Fig. 1.2.

The total power radiated by one meter squared of the surface of a black body – the intensity of radiation – can be calculated by integrating Eq. (1.1) over all wavelengths as

$$M^{\mathrm{e}} = \int_0^{\infty} M^{\mathrm{e}}_{\lambda}(T, \lambda)\mathrm{d}\lambda = \int_0^{\infty} \frac{2\pi hc^2}{\lambda^5(\mathrm{e}^{hc/\lambda k_{\mathrm{B}}T} - 1)}\mathrm{d}\lambda = \left| \begin{array}{l} u = hc/\lambda k_{\mathrm{B}}T \\ \mathrm{d}u = -hc\mathrm{d}\lambda/\lambda^2 k_{\mathrm{B}}T \\ 0 \to \infty,\ \infty \to 0 \end{array} \right| = \frac{2\pi k_{\mathrm{B}}^4 T^4}{h^3c^2} \int_0^{\infty} \frac{u\mathrm{d}u}{\mathrm{e}^u - 1}, \quad (1.2)$$

where

$$\int_0^{\infty} \frac{u\mathrm{d}u}{\mathrm{e}^u - 1} = \frac{\pi^4}{15}$$

is a known integral, we get

$$M^{\mathrm{e}} = \frac{2\pi^5 k_{\mathrm{B}}^4}{15h^3c^2}T^4 = \sigma T^4, \quad (1.3)$$

where

$$\sigma = \frac{2\pi^5 k_{\mathrm{B}}^4}{15h^3c^2} = 5.670\,374\ldots \times 10^{-8}\,\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$$

is the so-called Stefan-Boltzmann constant[2]. Formula (1.3) represents the Stefan-Boltzmann's law[3] and it says that the black-body radiation intensity is proportional to the fourth power of its thermodynamic temperature. Obviously, the total power P radiated by a black body with the surface-area S can be calculated as

$$P = S\sigma T^4. \quad (1.4)$$

[2]The value of the Stefan-Boltzmann constant is given by definition, it is not determined by measurement.

[3]This formula was first found experimentally in 1870 by the Austrian physicist Josef Stefan. Five years later it was derived by Austrian physicist Ludwig Boltzmann using general thermodynamics and Maxwell's theory.

From Fig. 1.2 it can be seen that the spectral-intensity curves have a maximum for the wavelength $\lambda_{\max}$, which is a function of the temperature. We find it by the following way.

If we introduce in Eq. (1.1) mew parameters $c_1 = 2\pi hc^2$, $c_2 = hc/k_B T$, taking the derivative, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{c_1}{\lambda^5(\mathrm{e}^{c_2/\lambda}-1)}\right] = -\frac{c_1}{c_2}\frac{(5\lambda/c_2-1)\mathrm{e}^{c_2/\lambda}-5\lambda/c_2}{\lambda^7(\mathrm{e}^{c_2/\lambda}-1)^2}. \tag{1.5}$$

A necessary condition for the extreme is zero derivative. If we introduce $u = c_2/\lambda$, the relation (1.5) will be zero if

$$5 - u = 5\mathrm{e}^{-u}. \tag{1.6}$$

Equation (1.6) is a transcendent equation that must be solved numerically. For example, in Maple, using the **fsolve** command, we can easily find that equation (1.6) has the non-zero solution $u = u_0 = 4.965114232\ldots$

The condition for the wavelength $\lambda_{\max}$ at which a black body radiates with a maximum is obtained by back-setting

$$u_0 = \frac{c_2}{\lambda_{\max}} = \frac{hc}{k_B T\lambda_{\max}} \quad\Rightarrow\quad \lambda_{\max} = \frac{hc}{u_0 k_B T} = \frac{b}{T}, \tag{1.7}$$

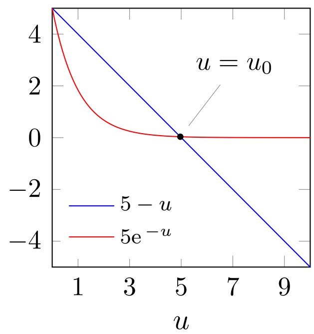


Figure 1.3: Solution of Eq. (1.6).

where

$$b = \frac{hcu_0}{k_B} = 2.897\,772\ldots\times 10^{-3}\,\mathrm{m}\cdot\mathrm{K}$$

is the so-called Wien constant[4]. Formula (1.7) represents the Wien's displacement law[5] and it says that the wavelength at which a black body radiates the most is inversely proportional to its thermodynamic temperature.

1.2.2 Real radiators

If the source of radiation is not a black body but a real body, the radiation is emitted according to the law

$$M'^{\mathrm{e}} = \int_0^\infty \epsilon(\lambda) M^{\mathrm{e}}_\lambda(T,\lambda)\mathrm{d}\lambda, \tag{1.8}$$

where $0 \le \epsilon(\lambda) \le 1$ is the so-called spectral emissivity of the surface of the radiator. If its value is independent on the wavelength. i.e., $\epsilon(\lambda) = \epsilon = const.$, we speak about the so-called grey body. The Stefan-Boltzmann's law has in this case the form

$$M'^{\mathrm{e}} = \epsilon\sigma T^4, \tag{1.9}$$

and for the total radiated power it holds

$$P = \epsilon S\sigma T^4. \tag{1.10}$$

1.3 Experiment

1.3.1 Experimental set-up

The aim of the experiment is to check the validity of the Stefan-Boltzmann's law (1.3) – to see that the power radiated by a black body is proportional to the fourth power of its thermodynamic temperature.

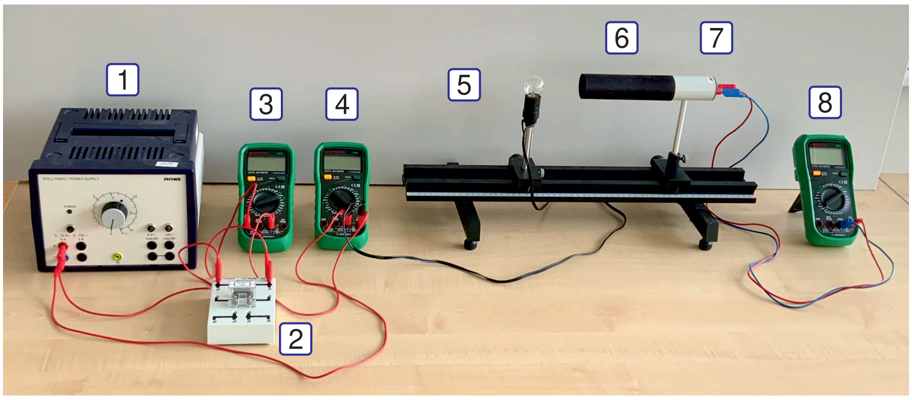
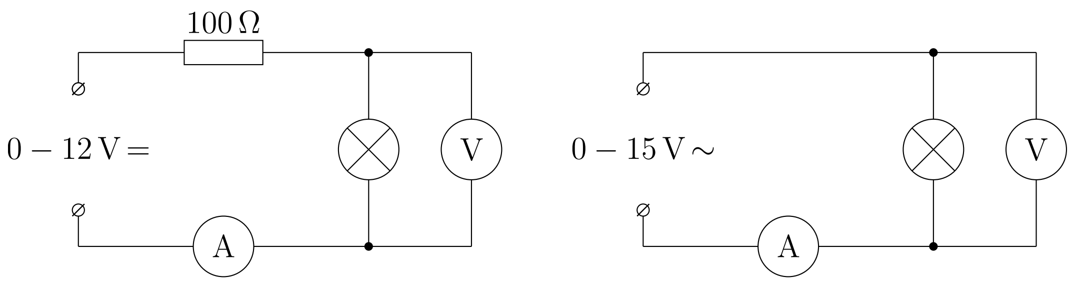


Figure 1.4: Experimental set-up: 1 – Supply of adjustable AC/DC voltage, 2 – jig with a 100 Ω ballast resistor, 3 – ammeter, 4 – voltmeter, 5 – light bulb, 6 – shielding tube, 7 – thermopile, 8 – millivoltmeter.



(a) (b)

Figure 1.5: Wiring diagram of the experiment.

The experimental set-up is shown in Fig. 1.4. A low-voltage tungsten-filament light bulb is used as the radiation source, representing here a grey body. The bulb is connected to a DC and AC power supply, see Fig. 1.5, the current flowing through the bulb and the voltage drop across the bulb are measured, so that the resistance of its filament can be calculated from Ohm's law as $R = U/I$.

The filament of the bulb is made of tungsten, for which the temperature dependence of the resistivity is known. For the resistance of the filament of the bulb, there is an empirical relationship which reads

$$R(t) = R_0(1 + \alpha t + \beta t^2), \tag{1.11}$$

where $\alpha = 4.82 \times 10^{-3}\,\mathrm{K}^{-1}$, $\beta = 6.76 \times 10^{-7}\,\mathrm{K}^{-2}$ are material parameters for tungsten, t is the temperature in degrees Celsius, and R_0 is the resistance of the filament at the temperature of 0° C. This resistance can be determined from Eq. (1.11) as follows. For known lab temperature t_{lab} the resistance of the filament $R(t_{\mathrm{lab}})$ is measured employing a small current (up to 100 mA), which does not cause heating of the filament. The resistance R_0 is then calculated as

$$R_0 = \frac{R(t_{\mathrm{lab}})}{1 + \alpha t_{\mathrm{lab}} + \beta t_{\mathrm{lab}}^2}. \tag{1.12}$$

When a larger current is passed through the filament, the filament of the bulb gets hotter, and depending on the temperature, its resistance changes to $R(t)$. Knowing the values of R_0, α, β, the

[4] The value of the Wien constant is given by definition, it is not determined by measurement.

[5] This law was theoretically derived in 1893 by German physicist Wilhelm Wien.

filament temperature can be calculated as the positive root of the quadratic equation (1.11), for which it holds

$$T = 273.15 + \frac{\alpha}{2\beta}\left[\sqrt{1+\frac{4\beta}{\alpha^2}\left(\frac{R(t)}{R_0}-1\right)}-1\right], \tag{1.13}$$

where T is the thermodynamic temperature (in kelvins).

In the experiment, a non-contact temperature sensor called the thermopile is directed against the filament of the bulb.[6] Voltage u_t between its output terminals is proportional to the power of the electromagnetic radiation impinging upon its surface. As its power is proportional to the total power radiated by the filament (1.10), it holds

$$u_\mathrm{t} \sim T^4. \tag{1.14}$$

The voltage u_t between the output terminals of the thermopile is measured employing a millivoltmeter.

1.3.2 Data processing and evaluation

By repeated measurements, for different voltages on the bulb, a set of pairs of values

$$[T_i, u_{\mathrm{t}i}], \qquad i = 1, 2, \ldots, N,$$

is obtained, which should follow the functional dependence (1.14). The measured data can be approximated employing the least squares method[7] by function

$$u_\mathrm{t} = AT^a, \tag{1.15}$$

where A is a constant dependent on the parameters of the experimental set-up, and the exponent a should have the value close to number 4. As the least squares method together with the relationship (1.15) leads to a set of nonlinear equations (which are hard to solve), the procedure is as follows. Equation (1.15) is logarithmed, resulting in

$$\log(u_\mathrm{t}) = \log\left(AT^a\right) = a\log(T) + \log(A) = a\log(T) + b, \tag{1.16}$$

where $b = \log(A)$, which is an equation for a line for logarithms of temperature and voltage. Value of the exponent a can be thus determined such that the pairs of logarithms of the measured data $[\log(T_i), \log(u_{\mathrm{t}i})]$ are approximated with a line (1.16), where the exponent a is its slope.

1.3.3 Procedure

1. Check that the filament of the bulb is positioned on the measuring bench on the axis of the thermopile and the shielding tube. If this is not the case, adjust the height of the bulb or the height and orientation of the thermopile accordingly.

2. Read the temperature t_lab displayed by thermometer.

3. Connect the bulb to the DC power supply terminals as shown in Fig. 1.5 (a). The 100 Ω ballast resistor allows for fine adjustment of small current through the bulb.

[6] It consists of a matrix of thermocouples connected in series.

[7] For this purpose, you can use an implementation of this method– *An universal tool for plotting graphs - least squares method* available at *http://planck.fel.cvut.cz/praktikum/*.

4. Switch the multimeter to the DC range 200 mV to measure the voltage on the thermopile. Switch the multimeters in the bulb circuit to DC ranges. Connect the ammeter to the **mA** terminal.

5. Measure the voltage drop across the bulb for currents ca. $25, 50, 75$, and 100 mA, calculate the resistance of the filament $R(t_{\mathrm{lab}})$ to see that for such small currents the filament resistance does not depend on the value of the current [8].

6. Remove (short-circuit) the $100\,\Omega$ ballast resistor, connect the circuit with the bulb to the AC voltage terminals of the power supply, see Fig. 1.5(b). Switch the multimeters connected to the bulb circuit to AC ranges, connect the ammeter to the **10A** terminal, measure the current using the 10 A$\sim$ range.

7. Using the adjustable power supply, gradually set the voltage drop across the bulb to 1 V, 1.5 V,2 V, . . . , 5.5 V, 6 V. After setting the voltage, wait for at least 5 minutes, until the thermopile output voltage stops increasing its value. Read the voltage drop across the bulb, the current flowing through it, and the voltage on the thermopile. **The voltage drop across the bulb should not exceed 6 V.**

8. For the individual voltages on the bulb, calculate the resistance of the filament and its temperature (see the text above).

9. Employing the least squares method (see the text above), calculate the value of the exponent in the Stefan-Boltzmann's law and its uncertainty, and compare the measured value with the theoretical value.

10. Plot the graph of $\log(u_{\mathrm{t}})$ as a function of $\log(T)$, showing the measured data and the approximating line (obtained by the least squares method).

11. Finally, in addition to the result of the measurement, answer the following questions. Is the bulb a linear circuit element? What was the maximum temperature reached by the filament of the bulb during your measurement? What wavelength of maximum emission corresponds to this temperature? **Bonus question:** Could you calculate what percentage of the energy the bulb emits, at the highest temperature you reached, in the visible part of the spectrum (400 nm – 800 nm)?

1.4 References

1. Jiří Bajer: Optika 1, vydavatelství Vladimír Chlup, 2018.

2. Raymond A. Serway, Clement J. Moses, Curt A. Moyer: Modern Physics, Thomson Learning, Inc., 2005.

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[8] If higher current was flowing through the bulb before, you need to wait for a while until the bulb filament gets cooled down.