

Laboratory experiment

Measurement of viscosity of liquids by the Stokes method

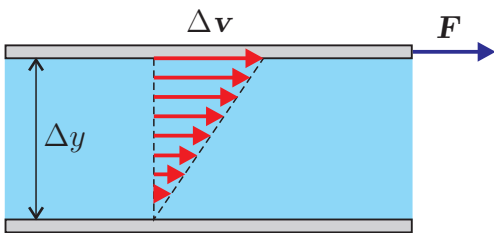
1.1 Task

Determine dynamic viscosity of glycerine and castor oil employing the Stokes viscosimeter.

1.2 Viscosity

A common property of liquids that distinguishes them from solids is their fluidity. They can flow from place to place, we can deform, divide, pour and strain them as we wish. They have no shape of their own, they adapt to the shape of the container, in which they also create a surface.

However, not all liquids are equally liquid, water or beer is certainly more liquid than oil, honey or Bailey's. The fluidity of a liquid is determined by its viscosity. The cause of viscosity is the tangential forces of internal friction, which causes the exchange of momentum between the molecules of adjacent layers of liquid, which move at different speeds.



If there is a viscous fluid between two parallel plates and the plates move with different velocities, a force must be applied to maintain a constant velocity. From the observations it can be concluded that the liquid acts against the mutual motion of the plates by the frictional force for the magnitude of which it applies

$$F = \eta S \frac{\Delta v}{\Delta y}, \quad (1.1)$$

where Δv is the mutual velocity of the plates, Δy is their separation distance, and S is the area. The proportionality coefficient η is called the coefficient of dynamic viscosity of a given liquid. Its unit is the pascal-second (Pa s). The Newton's formula (1.1) only holds for small mutual velocities¹.

The frictional force also acts between the individual layers of fluid if they move with different velocities relative to each other. In many cases, the velocity distribution in the flowing fluid is more complicated than in the previous case. In these cases, it is advantageous to use the formula (1.1) in a more general (differential) form

$$\tau = \eta \frac{dv}{dy}, \quad (1.2)$$

¹More precisely, for Newtonian fluids, Eq. (1.1) holds if $\rho \Delta v \Delta y / \eta < 1500$, where ρ is the liquid density. When this condition is met, the velocity gradient is constant.

where $\tau = F/S$ is the shear stress. The liquids for which the tangential stress is proportional to the velocity gradient and the relation (1.2) applies to them, are called the Newtonian liquids. These include, for example, water, mineral oils, alcohol, etc. Even real gases have a low viscosity, which is the cause of aerodynamic drag. The viscosity of fluids is strongly temperature-dependent. While it decreases with temperature for liquids, it increases with temperature for gases.

In the case of so-called non-Newtonian fluids, the dependence of the tangential stress and the velocity gradient is not linear, as described by the formula (1.2). These include, for example, varnishes, sludge, crude oil, lime, chocolate, and others.

Liquid	η [mPa s]	Gas	η [μ Pa s]
Glycerine	1480	Neon	32.1
Castor oil	989	Oxygen	20.8
Olive oil	80.8	Helium	20.0
Sulfuric acid	25.4	Air	18.6
Mercury	1.554	Hydrogen	9.0
Water	1.002	Propane	8.3

Table 1.1: Dynamic viscosity of some liquids and gases for $T = 20^\circ\text{C}$ and normal atmospheric pressure.

1.3 Drag

Viscosity is always the cause of fluid resistance to moving objects—the drag. The so-called d’Alembert’s paradox can be proved, according to which an ideal fluid with zero viscosity does not put any drag to bodies flown around. When a body moves in a viscous fluid, it is subjected to a resistive force, the direction of which is opposite to the direction of the motion of the body. During slow motion, we assume that the viscous fluid adheres to the walls of the body, the shear stress at these walls is not zero and therefore the fluid acts on the body with a non-zero drag. Another source of drag acting on the body is the unevenly distributed pressure in front of and behind the body (this unevenness is also caused by the viscosity), the spatial variation of which is related to the nature of the flow around the body.

The drag that the environment puts at a moving (flown around) body has a relatively complex functional dependence of the body shape, speed and viscosity, and in most cases it is determined experimentally. An important parameter here is the dimensionless Reynolds number, defined as

$$\text{Re} = \frac{vl}{\nu},$$

where v is the mutual velocity of the body and liquid², l is a characteristic dimension of the object (for example, a diameter for a sphere) and $\nu = \eta/\rho$ is the kinematic viscosity, where ρ is the liquid density.

If $\text{Re} < 1$ applies to the Reynolds number, the influence of viscous shear forces prevails in the drag. For a sphere, in this case, Stokes’ law can be derived for the magnitude of the drag, which reads

$$F = 6\pi\eta r v, \tag{1.3}$$

²The mutual velocity is measured relative to a place, where the velocity of the liquid is not disturbed by the presence of the moving object.

where r is the sphere radius. The Stokes' law thus describes the fact that the magnitude of the drag for a small Reynolds number is proportional to the magnitude of the velocity of the body.

In general, Newton's formula is used to describe the drag for different body shapes and different velocities, which reads

$$F = \frac{1}{2}C\rho Sv^2, \quad (1.4)$$

where $C = C(\text{Re})$ is the drag coefficient and S is the cross-sectional area (with respect to the mutual velocity) of the body. The drag coefficient depends on the shape of the body and, in general, on the Reynolds number, and this functional dependence is determined experimentally. For example, for a sphere and the Reynolds number in the interval $10^3 < \text{Re} < 10^5$ it holds $C \approx 0.5$. Comparing the formulas (1.3) and (1.4) we can write for a sphere and $\text{Re} < 1$ that

$$C = \frac{12\eta}{\rho rv} = \frac{24}{\text{Re}}.$$

1.4 Free fall in a viscous liquid

Determination of the viscosity of a liquid using the Stokes viscometer is based on the study of the free fall of a ball in the investigated liquid. There are three forces acting on the ball: gravity \mathbf{F}_g , buoyancy \mathbf{F}_b and drag \mathbf{F}_d . For the gravity, it reads

$$\mathbf{F}_g = m\mathbf{g},$$

where m is the mass of the ball and \mathbf{g} is the gravitational acceleration. According to Archimedes' law, the magnitude of the buoyancy is equal to the weight of the liquid, which would occupy the volume of the immersed body, so it reads

$$\mathbf{F}_b = -m_{\text{liquid}}\mathbf{g} = -\frac{4}{3}\pi r^3\rho\mathbf{g},$$

where r is the ball radius. If the viscosity of the liquid is large and at the same time the radius of the ball is small, the drag for $\text{Re} < 1$ will follow the Stokes' law and we can write

$$\mathbf{F}_d = -6\pi\eta r\mathbf{v},$$

where \mathbf{v} is the ball velocity.

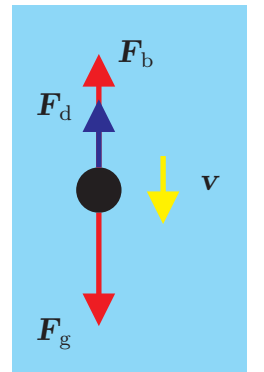
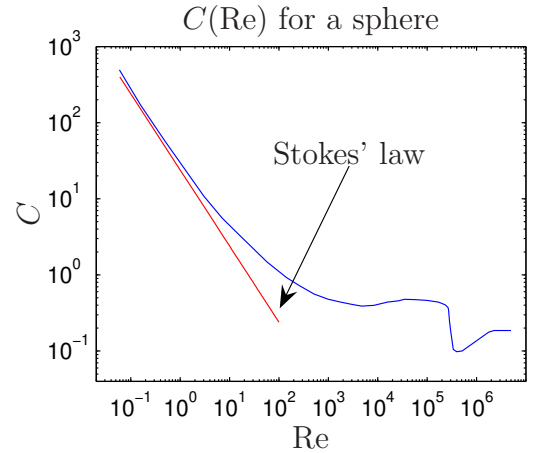
The equation of motion for the ball can therefore be written in the form

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}_g + \mathbf{F}_b + \mathbf{F}_d = \left(m - \frac{4}{3}\pi r^3\rho\right)\mathbf{g} - 6\pi\eta r\mathbf{v}. \quad (1.5)$$

The vector equation (1.5) can be further simplified. If the initial ball velocity is zero, $\mathbf{v}(t=0) = \mathbf{v}_0 = \mathbf{0}$, it will move in a straight line in the direction of the gravitational acceleration vector³. If we choose the direction of the gravitational acceleration vector as positive, we can reduce the equation (1.5) for the velocity vector component in the respective direction to

$$m\frac{dv}{dt} = \left(m - \frac{4}{3}\pi r^3\rho\right)g - 6\pi\eta rv. \quad (1.6)$$

³That is, if its average density is greater than the density of the fluid.



If we introduce auxiliary quantities $\alpha = g(1 - \rho/\rho_b)$, where ρ_b is the ball average density and $\beta = 9\eta/(2r^2\rho_b)$, Eq. (1.6) can be rewritten as

$$\frac{dv}{dt} = \alpha - \beta v \quad \Rightarrow \quad \frac{dv}{\alpha - \beta v} = dt \quad \Rightarrow \quad \int_0^v \frac{dx}{\alpha - \beta x} = \int_0^t dy$$

and the evaluation of the integrals results in

$$t = \left[-\frac{1}{\beta} \ln |\alpha - \beta x| \right]_0^v = \frac{1}{\beta} \ln \left| \frac{\alpha}{\alpha - \beta v} \right| \quad \Rightarrow \quad v = \frac{\alpha}{\beta} (1 - e^{-\beta t}) = v_\infty (1 - e^{-\beta t}). \quad (1.7)$$

From the last formula it follows that the velocity of the ball increases exponentially from zero to the value

$$v_\infty = \frac{2gr^2(\rho_b - \rho)}{9\eta}. \quad (1.8)$$

The time of reaching of p -th multiple⁴ of the velocity v_∞ can be calculated from Eq. (1.7), namely,

$$p = 1 - e^{-\beta t_p} \quad \Rightarrow \quad t_p = -\frac{\ln(1-p)}{\beta} = -\ln(1-p) \frac{2r^2\rho_b}{9\eta}. \quad (1.9)$$

During this time, the sphere travels the distance⁵

$$\begin{aligned} \Delta l_p &= \int_0^{t_p} v_\infty (1 - e^{-\beta t}) dt = v_\infty \left[t + \frac{1}{\beta} e^{-\beta t} \right]_0^{t_p} = v_\infty \left[t_p + \frac{1}{\beta} (e^{-\beta t_p} - 1) \right] = \\ &= -\frac{v_\infty}{\beta} [\ln(1-p) + p] = -[\ln(1-p) + p] \frac{4gr^4\rho_b(\rho_b - \rho)}{81\eta^2}. \end{aligned} \quad (1.10)$$

For example, for $p = 0,99$ we get

$$t_{99} \approx \frac{\rho_b r^2}{\eta}, \quad \Delta l_{99} = 0,18 \frac{gr^4\rho_b(\rho_b - \rho)}{\eta^2}. \quad (1.11)$$

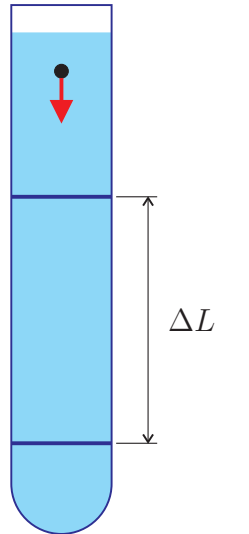
1.5 Stokes viscosimeter

The Stokes viscosimeter is a tall transparent vessel filled with the examined liquid in which the time of free fall ΔT of a suitably selected ball between two marks at a distance ΔL is measured. The viscosity is calculated using Eq. (1.8) rewritten in the form

$$\eta = \frac{2}{9} gr^2(\rho_b - \rho) \frac{\Delta T}{\Delta L} \quad (1.12)$$

if the following conditions are met.

- The marks are placed at such a distance that the motion of the ball between them can be considered uniform.
- The radius of the test vessel is significantly larger than the radius of the ball, it should not move near the wall.
- It holds $Re < 1$. Otherwise, a ball with a lower density and radius should be chosen.



⁴ $p \in (0, 1)$

⁵You can see for yourself that it holds $\ln(1-p) + p < 0$ for $p \in (0, 1)$.

The table below shows the values calculated for different liquids at temperature 20°C and an iron ball with radius 1 mm ($\rho_b = 7860 \text{ kg m}^{-3}$).

Liquid	η [mPa s]	ρ [kg m ⁻³]	v_∞ [cm s ⁻¹]	t_{99} [ms]	Δl_{99} [mm]	Re [-]
Glycerine	1480	1261	0.97	5.4	0.041	0.017
Castor oil	989	960	1.53	8.2	0.099	0.030
Transformer oil	866	31.6	48.2	254	96.4	26.4

It can be seen from the table that to measure the viscosity of the transformer oil, it would be necessary to proceed differently, because using Stokes' law, it yields $\text{Re} > 1$, but, for these values the Stokes' law does not hold true any more.

1.6 Procedure

The measurement procedure is the same for both glycerine and castor oil.

1. Count 12 balls in a Petri dish and measure their radius.
2. Determine the weight of the balls on the analytical balance by first weighing the Petri dish with the balls and then **the same** Petri dish without the balls.
3. On the test vessel with the examined liquid, set and measure the distance of the rubber rings, between which you will measure the time of fall of the balls. Place the upper ring at least 5 cm below the liquid surface.
4. Use a double plummet to check and, if necessary, adjust the vertical direction of the test vessel.
5. Use a stopwatch to measure the time the balls fall between the rubber rings. Discard the smallest and largest measured value.
6. Use a densitometer to read the density of the liquid.
7. Due to the strong temperature dependence of the viscosity, use a thermometer to read the temperature and state it together with the result of your measurement.

1.7 References

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3. Antonín Havránek: Klasická mechanika II - Kontinuum, *Univerzita Karlova v Praze*, Praha, 2003.
4. Jaromír Brož: Základy fyzikálních měření, *Státní pedagogické nakladatelství*, Praha, 1983.