Laboratory experiment

Determination of the Young's modulus

1.1 Task

1. Determine the Young's modulus for two sample materials and compare the measured results with tabulated values.

1.2 Theory

1.2.1 Mechanical stress

Unlike idealized *rigid bodies*, if external forces act on real bodies, they can cause a change in their volume, shape (bending, twisting, wrinkling, ...), possibly even breaking, rupture, etc. In these cases we speak of elasticity and strength of the bodies.

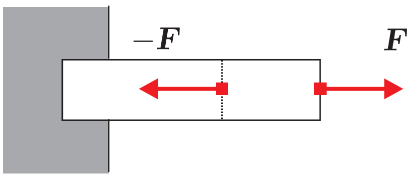

Figure 1.1: Mechanical stress.

Consider a rod that is firmly embedded in a wall at one end and is stretched at the other end by a tensile force $\boldsymbol{F}$, see Fig. 1.1. The rod is at rest and in equilibrium, but its state is different from the case when it is not stretched. The difference is in the presence of mechanical stress throughout the volume of the rod, which is due to the tensile force. The presence of this stress can be demonstrated as follows. The rod is cut by an imaginary cutting plane perpendicular to its axis. Since the cut portion of the rod stays at rest, a force $-\boldsymbol{F}$ must act upon the cut-away part to compensate for the tensile force $\boldsymbol{F}$. This force acts at any cutting plane of the rod and is mediated by the inter-atomic forces of the rod material. The measure of the mechanical stress of the material is the quantity

$$\sigma = \frac{F}{S},$$

where S is the cross-sectional area, which is called the mechanical stress and which is measured in pascals (Pa), the same way as the pressure in fluids.

In the previous thought experiment, we intuitively chose a cutting plane perpendicular to the longitudinal axis of the rod, so the normal vector of the cutting plane is collinear with the force reaction $-\boldsymbol{F}$. In this case we speak of the so-called normal stress. Since the plane of the imaginary cut can in general be chosen arbitrarily, the force reaction can have any angle with respect to the cut normal and the mechanical stress in this general case cannot be described by a scalar quantity, so it is generally a tensor quantity. In the following, however, by mechanical stress we shall mean the normal stress.

1.2.2 Hooke's law

If a rod of length l_0 is subjected to a longitudinal tensile force of magnitude F, it will be extended to length l. If the applied force is not too great, the elongation $\Delta l = l - l_0$ is directly proportional to its magnitude and Hooke's law holds

$$\Delta l \sim F.$$

After the tensile force is removed, the rod is shortened again back to its original length l_0. We are talking about the so-called *elastic deformation*. Experiments show that the elastic elongation of a rod is directly proportional to its length l_0 and inversely proportional to its cross-sectional area S_0, so that the Hooke's law can be written as

$$\Delta l = \frac{l_0}{ES_0}F, \tag{1.1}$$

where the proportionality constant E is called the Young's modulus of elasticity, it depends on the material of the rod (it is a material constant) and it is measured in pascals. Hooke's law also applies to deformation in compression, and both deformations are described by the same Young's modulus. From the formula (1.1) it is possible to eliminate the geometric dimensions of the rod by introducing the relative elongation (strain) of the rod $\varepsilon = \Delta l/l_0$ and the mechanical stress $\sigma = F/S_0$. The formula then takes the form

$$\varepsilon = \frac{\sigma}{E}. \tag{1.2}$$

1.2.3 Stress-strain curve

The above mentioned Hooke's law is valid only for *small* stresses, the dependence $\sigma(\varepsilon)$ is generally not linear, we graphically represent it in the so-called stress-strain curve, see Fig. 1.2.

Hooke's law is valid in the linear region of the stress-strain curve up to the so-called proportionality limit (point A). This is followed by a short region ending at the yield strength (point B). Up to this point, the rod returns to its original length l_0 when the applied force is released, although Hooke's law no longer applies. This is followed by an region of permanent deformation where the rod does not return to its original length after the applied force is released. This region contains the so-called ultimate strength (point C), where the relative elongation increases without further increase in stress due to creep of the rod material changing the internal structure. If the strain continues to increase, the rod ruptures (point D).

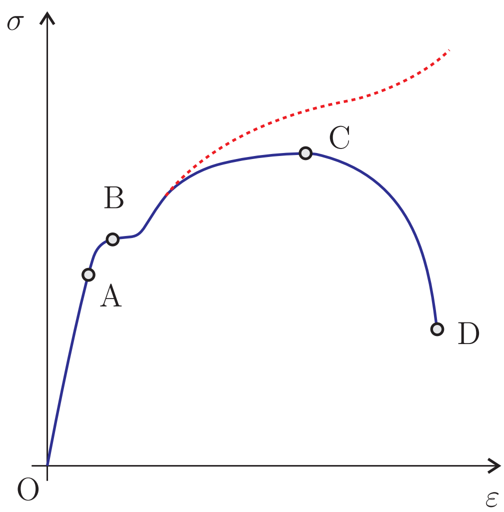

Figure 1.2: Stress-strain curve.

In the last region of the stress-strain curve (between points C and D) it seems that the mechanical stress decreases with increasing strain, but this is not true. The apparent decrease in stress is due to the shrinking of the cross-section S of the rod as it is stressed. Since it is practically impossible to measure the shrinking cross-section of the rod under tension, we relate the tensile force to the original cross-section of the rod S_0, thus causing this apparent decrease. The actual mechanical stress F/S is an increasing function, indicated by the dashed line in the figure.

1.2.4 Hysteresis

In the region of permanent deformation, the memory of the deformed material is usually manifested, which can be illustrated by the so-called hysteresis loop BCDEB, see Fig. 1.3.

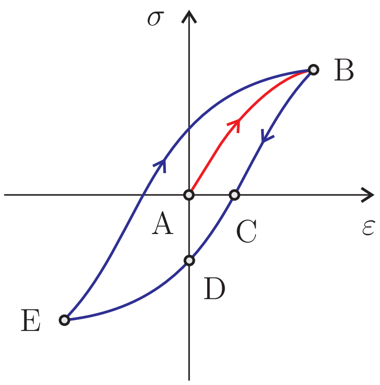

Figure 1.3: Hysteresis loop.

During the initial deformation of the material, the deformation follows the virgin curve AB, which is initially linear (Hooke's law). As the stress subsequently decreases, the deformation changes according to the BC curve, and when the stress is released, the permanent strain ε_C corresponding to the point C remains. The compressive stress σ_D (point D) is then required to remove it.

1.2.5 Change of the volume

Consider a rod in the form of a cube with dimensions $a \times b \times c$. Let there be a *small* tensile force of magnitude F along the side a. According to Hooke's law, the relative extension of the rod will be $\varepsilon = \Delta a/a = \sigma/E$. Precise measurements show that as the rod lengthens longitudinally, it also shrinks transversely; the relative narrowing $\eta = \Delta b/b = \Delta c/c$ is directly proportional to the longitudinal relative elongation and it holds

$$\eta = -k\varepsilon,$$

where k is the so-called Poisson's ratio, which is a material parameter. Thus, for the change in the volume of a rod subjected to a longitudinal tension, we get

$$\Delta V = (a + \Delta a)(b + \Delta b)(c + \Delta c) - abc \quad \Rightarrow \quad \frac{\Delta V}{V} \approx (1 - 2k)\varepsilon.$$

For usual engineering materials, the value of Poisson's ratio is usually in the interval $0.25 \sim 0.35$, so that the tensile stresses increase their volume. The material parameters E and k fully determine the elastic properties of a homogeneous isotropic material.

1.3 Experiment

1.3.1 Autocollimator

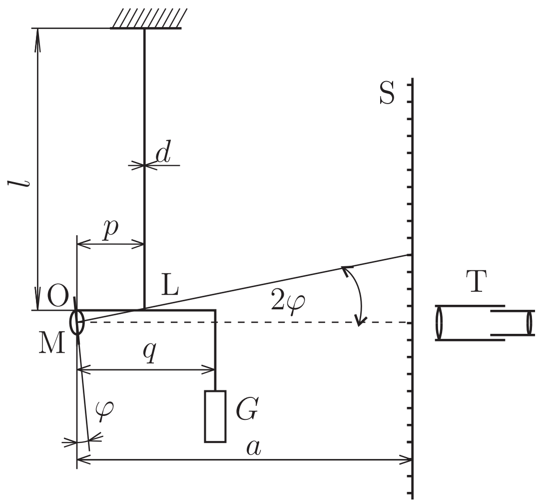

Figure 1.4: A simple autocollimator.

Since there are only very small length changes, the measurements of the magnitude of elongation of the material under tensile stress are made employing a simple autocollimator.

The measurement is carried out in such a way that the measured specimen (wire) of length l and diameter d is first loaded with a weight of mass m_0, suspended at the end of the lever L of length q (see Fig. 1.4), which has the purpose of straightening the wire.

The distance of the wire attachment from the axis of rotation O is p. On the axis of rotation of the lever L there is a mirror M in which the scale S reflects. The scale reflected in the mirror is observed through the telescope T. Hanging another weight on the end of lever L causes an increase in the gravitational force[1] by the value $\Delta G = \Delta mg$, where Δm is the mass of the added weight (and hence the change in mass loading the lever) and g is the acceleration due to gravity, the wire will elongate by Δl and as a result the mirror M will rotate by φ. According to the laws of geometrical optics, if a plane mirror is rotated by an angle φ, the reflected beam is deflected by double the angle.

The rotation of the mirror M is reflected in the telescope by the fact that instead of the scale division n_0, which was in the centre of the cross-hair of the telescope when the mass m_0 was loaded, the division n (one division corresponds to 1 mm) is moved to the centre of the cross-hair. Assuming that the wire lengthening of Δl is considerably smaller than the length of the wire l and the distance of the mirror from the scale a, we can calculate the angle φ as

$$\frac{\Delta n}{a} = \mathrm{tg}\ 2\varphi \approx 2\varphi \qquad \Rightarrow \qquad \varphi = \frac{\Delta n}{2a}\ , \tag{1.3}$$

where $\Delta n = n - n_0$. If the wire is lengthened by Δl, the lever L rotates by an angle φ and according to Fig. 1.4 it applies

$$\frac{\Delta l}{p} = \mathrm{tg}\ \varphi \approx \varphi. \tag{1.4}$$

Eliminating the angle φ from Eqs. (1.3) and (1.4) we obtain the relation for the strain ε in form

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{p\Delta n}{2al_0}\ . \tag{1.5}$$

Substitution into Eq. (1.1) for the strain from Eq. (1.5), and setting $F = q\Delta G/p$ and $S_0 = \pi d^2/4$, we get the formula for the determination of the Young's modulus E as

$$E = \frac{8gal_0q}{\pi d^2p^2}\frac{\Delta m}{\Delta n}\ . \tag{1.6}$$

In practice it is very difficult to determine the length of the unloaded wire as it may be (it is) twisted. Therefore, we measure the initial length l_0 with a pre-load of one weight[2] of mass m_0. A wire so pre-loaded (straightened) will be considered as "unloaded".

Adjust the telescope so that the scale and the cross-hairs are clearly visible. Set the distance of the mirror from the scale to 1 m. Depending on the material of the wire to be measured and its diameter, choose the mass of the weight and gradually add five pieces of the chosen weight. Then, gradually remove the weights up to to the base load, which is used to straighten the wire.

1.3.2 Data processing

We could evaluate the measurements tentatively by substituting the geometric dimensions, the acceleration due to the gravity to the relation (1.6) together with $\Delta m = 5 \times m_1$, where m_1 is the mass of one weight, and $\Delta n = \Delta n_5$, which is the corresponding displacement of the cross-hair on the scale. However, we will proceed differently to use all the measured data to verify that the adopted assumptions have been met.

Equation (1.6) can be rewritten into the form

$$\Delta n = \alpha \Delta m, \tag{1.7}$$

[1] Which, however, acts on the wire through the lever mechanism.

[2] Even if we load the wire with this weight and it is extended by $\Delta l'$ and we do not actually measure the length of l_0, but $l_0 + \Delta l'$, $\Delta l' \ll l_0$, so we can take the measured length as the length l_0 of the unloaded wire. The corresponding lengthening of $\Delta l'$ is safely masked by the uncertainty of the wire length measurement by the tape measure.

where

$$\alpha = \frac{8gal_0q}{\pi d^2p^2E} \tag{1.8}$$

is the proportionality constant. The value pairs $[\Delta m_i, \Delta n_i]$, where $\Delta m_i = im_1$ are plotted in a graph. The data points should be distributed along a straight line passing through the origin. Employing the least squares method[3] the data points are approximated by a first-degree polynomial

$$\Delta n = A\Delta m + B, \tag{1.9}$$

resulting in the parameter values A and B (and their uncertainties). Comparing the relations (1.9) and (1.7) and substituting into Eq. (1.8) we get the formula for the calculation of the Young's modulus

$$E = \frac{8gal_0q}{\pi d^2p^2A},$$

and its uncertainty is calculated by the usual way.

1.3.3 Procedure

Caution: Before starting the measurement, it is necessary to put on a protecting face shield.

1. Measure the length of the wire at the basic load with a metal tape measure[4], measure the distance between the mirror and the scale.

2. Use the caliper to measure (several times and as accurately as possible) the length of the lever q and the distance p of the wire attachment on the lever L from the axis of rotation O.

3. Measure (at least 10×) the diameter of the wire with a micrometer at various points to verify that the wire diameter is constant.

4. Select the mass of the weights according to the material and diameter of the wire. Use a set of half-kilogram weights for the thinner wire and a set of one-kilogram weights for the thicker wire.

5. Using the telescope, read the "zero" value of n'_0 on the scale, add the individual weights and subtract the corresponding values of n'_i where $i = 1, \ldots, 5$ to calculate $\Delta n'_i = n'_i - n'_0$.

6. Denote $n''_5 = n'_5$, remove the weights one by one and subtract the values n''_i, where $i = 4, \ldots, 0$ in the telescope to calculate $\Delta n''_i = n''_i - n''_0$. After the measurement, only the base weight is hung on the apparatus to straighten the wire.

7. Calculate $\Delta n_i = (\Delta n'_i + \Delta n''_i)/2$.

8. Plot a graph of the data points $[\Delta m_i, \Delta n_i]$ approximated by a straight line employing the least squares method, see the paragraph 1.3.2.

[3] To do this, you can use the Universal tool for plotting graphs at *http://planck.fel.cvut.cz/praktikum/*. It will occur to the careful reader that the measured data could be approximated by a straight line passing through the origin, since determining the coefficient B (which should be zero) unnecessarily deprives us of one degree of freedom. The situation is usually such that the wire is still slightly twisted even at the base load and the value of Δn_1 is usually burdened with some systematic error, which is compensated by the procedure used.

[4] The measured wire passes through the retaining screw and therefore its length is measured from the head of this screw

9. From the measured data, calculate the Young's modulus (and its uncertainty), see the paragraph 1.3.2.

10. Repeat the measurement for the second specimen (wire).

1.4 References

1. Michal Bednařík, Petr Koníček, Ondřej Jiříček: Fyzika I a II – Fyzikální praktikum, *[skriptum], Vydavatelství ČVUT*, Praha, 2003.

2. Jiří Bajer: Mechanika 3, *Univerzita Palackého v Olomouci*, Olomouc, 2012.

3. Richard P. Feynman, Robert B. Leighton a Matthew Sands: Feynmanovy přednášky z fyziky 2/3 s řešenými příklady, *Fragment*, Praha, 2001.

1.5 Appendix – Selected properties of some materials

Material	E [10^{10} Pa]	G [10^{10} Pa]	k -
Aluminium	7.07	2.64	0.34
Copper	12.3	4.55	0.35
Lead	1.6	0.56	0.44
Diamond	112	52	0.1
Zinc	9.0	3.6	0.25
Iron α	21.2	8.2	0.29
Steel	20-21	7,9-8,9	0,25-0,33
Steel (1% C)	21.0	8.1	0.29
Welding steel	20.4	7.9	0.29
Bronze	9.7-10.2	3.3-3.7	0.34-0.40
Phosphor bronze	12.0	4.36	0.38
Brass	9.9	4.2	0.37
Duralumin	7.25	2.75	0.34
Plexiglass	0.33	0.12	0.35

Table 1.1: Young's modulus E, shear modulus G, and Poisson's ratio k for selected materials at room temperature.

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(úprava původního textu [1])